Identifying Subspaces from Canonical Projections

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Applied Algebra Seminar
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Robert Nowak and Nigel Boston

Outline

- Introduction
- Problem Description
- Setup
- The Answer
- Sketch of the proof
- Application
- Conclusions

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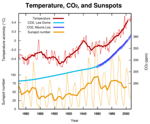

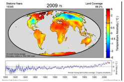

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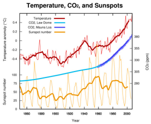

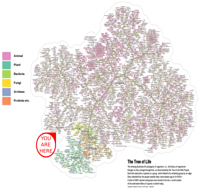

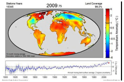

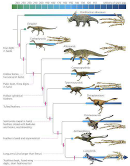

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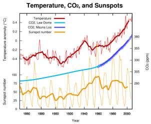

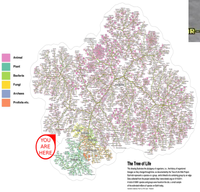

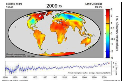

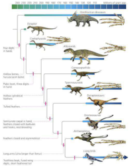

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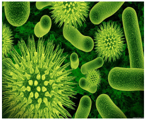

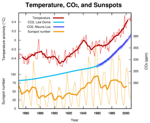

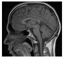

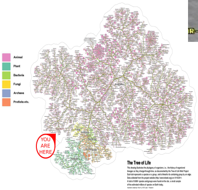

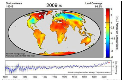

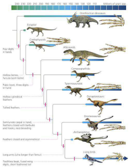

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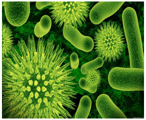

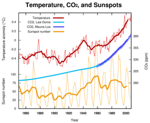

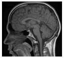

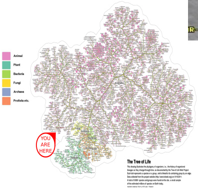

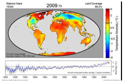

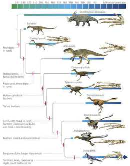

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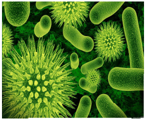

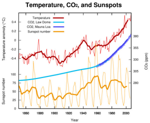

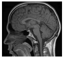

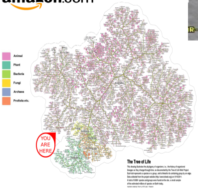

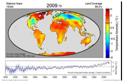

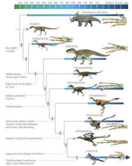

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movies \left\{ \underbrace{\begin{bmatrix}
4 & \cdot & 1 & 1 & 5 & 4 & 2 & 5 & 3 & \cdot \\
\cdot & 5 & \cdot & 1 & \cdot & \cdot & \cdot & 4 & \cdot & 5 \\
1 & \cdot & 2 & \cdot & 2 & 5 & 1 & \cdot & \cdot & 4 \\
2 & \cdot & 5 & 4 & \cdot & 1 & \cdot & 2 & \cdot & 4 \\
\cdot & 5 & 1 & \cdot & 5 & \cdot & \cdot & \cdot & 1 & \cdot \\
1 & 5 & \cdot & 2 & \cdot & 2 & 1 & \cdot & \cdot & 5
\end{bmatrix}}_{people} \right.
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\cdot & 5 & 1 & \cdot & 5 & \cdot & \cdot & \cdot & 1 & \cdot \\
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Nobody has seen every movie.

Introduction

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movies \left\{ \underbrace{\begin{bmatrix} 4 & 5 & 1 & 1 & 5 & 4 & 2 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 & 1 & 4 & 4 & 3 & 5 \\ 1 & 3 & 2 & 5 & 2 & 5 & 1 & 3 & 2 & 4 \\ 2 & 1 & 5 & 4 & 3 & 1 & 5 & 2 & 5 & 4 \\ 3 & 5 & 1 & 1 & 5 & 1 & 4 & 3 & 1 & 3 \\ 1 & 5 & 3 & 2 & 2 & 2 & 1 & 4 & 5 & 5 \end{bmatrix}}_{people} \right.
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We would be able to make good recommendations!

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$$clothes \left\{ \underbrace{\begin{bmatrix} 4 & \cdot & 1 & 1 & 5 & 4 & 2 & 5 & 3 & \cdot \\ \cdot & 5 & \cdot & 1 & \cdot & \cdot & \cdot & 4 & \cdot & 5 \\ 1 & \cdot & 2 & \cdot & 2 & 5 & 1 & \cdot & \cdot & 4 \\ 2 & \cdot & 5 & 4 & \cdot & 1 & \cdot & 2 & \cdot & 4 \\ \cdot & 5 & 1 & \cdot & 5 & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & 5 & \cdot & 2 & \cdot & 2 & 1 & \cdot & \cdot & 5 \end{bmatrix}}_{people} \right.$$

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- Linear algebra is one of our favorite tools...
  - We know lots about linear algebra.
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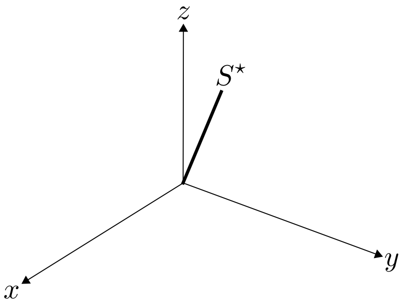
- There is great interest on extending usage of linear algebra to incomplete datasets.
- That is what we are studying.

Outline

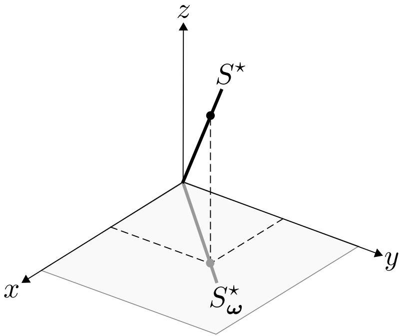
Problem description

$S^\star :=$ r-dimensional subspace of $\mathbb{R}^d$, $r < d$.



Problem description

$S^\star_{\boldsymbol{\omega}} :=$ Projection of $S^\star$ onto a canonical subspace.



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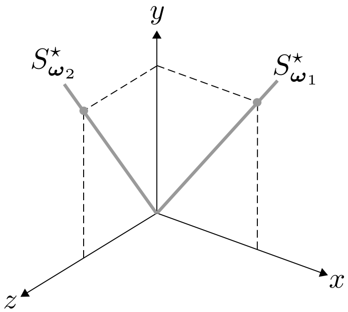

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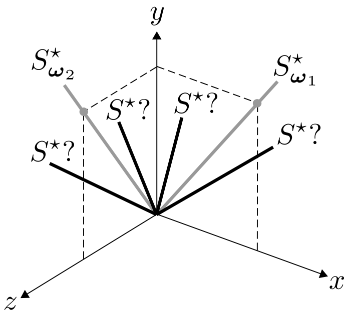
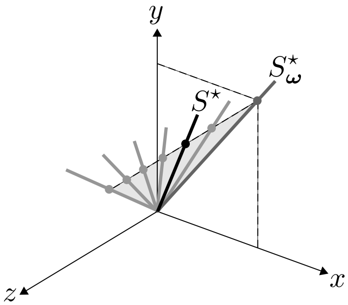


Can you uniquely determine $S^\star$ from this set of projections?

Problem description

Is this even possible?

There might be many subspaces that agree with the projections.



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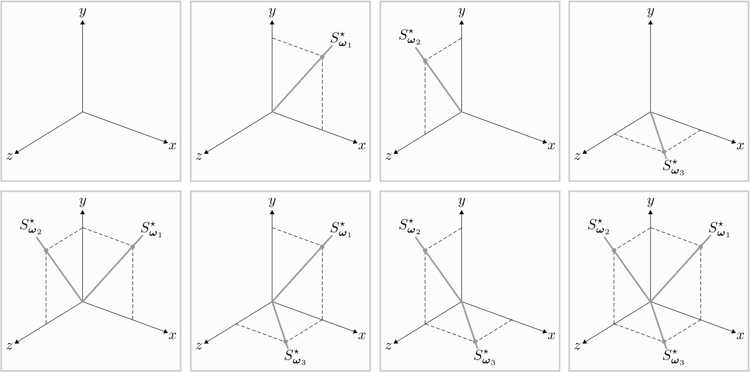
Well... it depends on which set of projections I give you.



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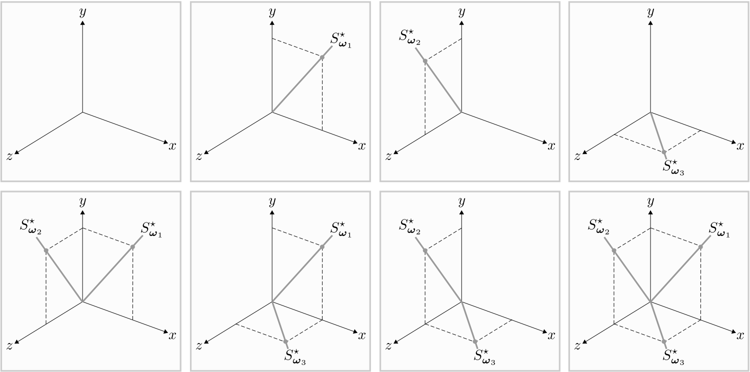

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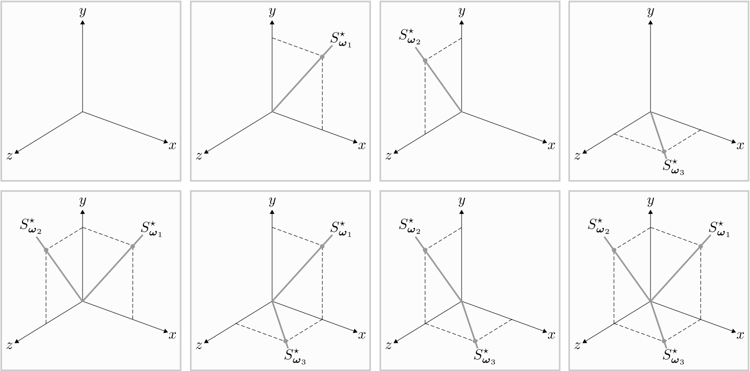

Can you tell which are *the good sets*?

This is what we answer here: which are *the good sets*.

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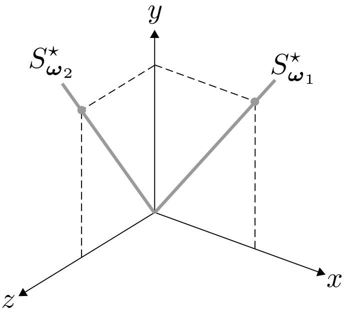
Setup

The columns of $\boldsymbol{\Omega}$ will index the given projections.



$$\boldsymbol{\Omega} = \begin{array}{c} \begin{array}{cc} \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{array}\right] \end{array}$$

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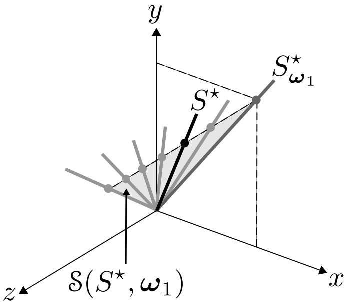

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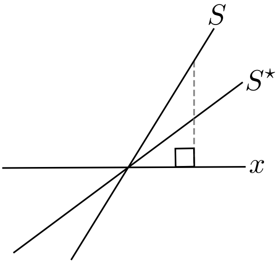

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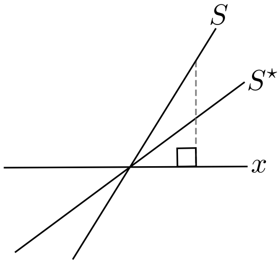

- $\Rightarrow$ Assume w.l.o.g. that all projections are onto $r + 1$ canonical coordinates.

Setup

- For any matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$:

$$\mathbf{\Omega}' = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{n(\mathbf{\Omega}') \;:=\; \#\text{columns}} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m(\mathbf{\Omega}') := \#\text{nonzero rows}$$

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- $d - r$ projections are *necessary*, so we will assume w.l.o.g.

$$n(\mathbf{\Omega}) = d - r.$$

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The Answer

Theorem (Pimentel-Alarcón, Nowak, Boston, '14)

For almost every $S^\star$*, with respect to the uniform measure over* $\mathrm{Gr}(r, \mathbb{R}^d)$*,* $S^\star$ *is the only subspace in* $\mathcal{S}(S^\star, \mathbf{\Omega})$ *if and only if for every matrix* $\mathbf{\Omega}'$ *formed with a subset of the columns in* $\mathbf{\Omega}$*,*

$$m(\mathbf{\Omega}') \;\geq\; n(\mathbf{\Omega}') + r.$$

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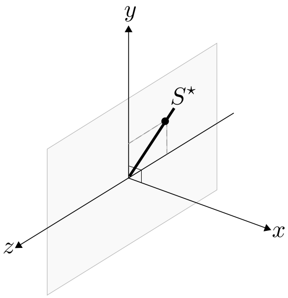

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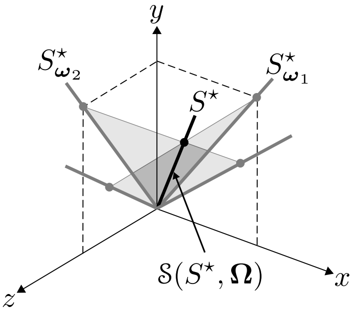
This is what we want!



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For almost every $S^\star$, with respect to the uniform measure over $\mathrm{Gr}(r, \mathbb{R}^d)$, $S^\star$ is the only subspace in $\mathcal{S}(S^\star, \mathbf{\Omega})$ if and only if for every matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$,

$$m(\mathbf{\Omega}') \;\geq\; n(\mathbf{\Omega}') + r.$$

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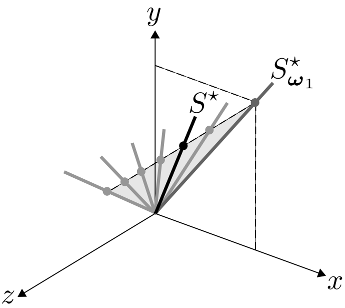
$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\Rightarrow\; \text{Check: } \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

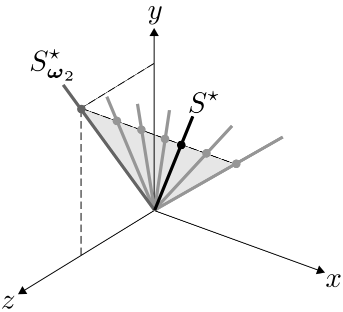
Outline

- Introduction
- Problem Description ✓
- Setup ✓
- The Answer ✓
- Sketch of the proof
- Application
- Conclusions

Sketch of the proof

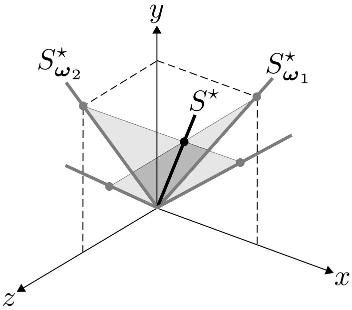
We will find the subspaces that agree with *each* projection.





Sketch of the proof

Then find the intersection.



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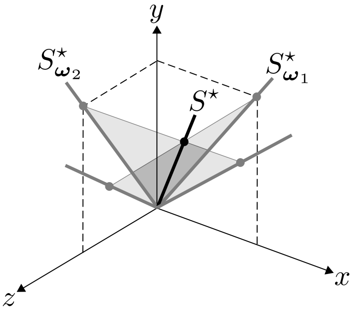
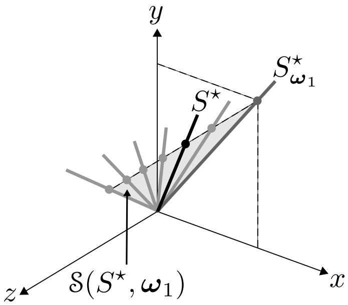


If the intersection only contains one subspace, then ;)

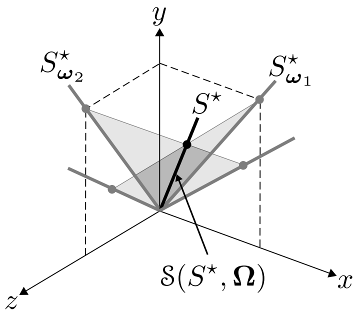
Sketch of the proof

$\mathcal{S}(S^\star, \boldsymbol{\omega}_i) :=$ Set of r-dimensional subspaces matching $S^\star$ on $\boldsymbol{\omega}_i$.



Sketch of the proof

$$\mathcal{S}(S^\star, \boldsymbol{\Omega}) = \bigcap_i \mathcal{S}(S^\star, \boldsymbol{\omega}_i).$$



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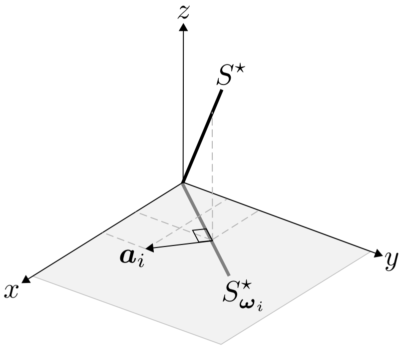
$\boldsymbol{a}_i :=$ Vector orthogonal to the i^{th} projection.



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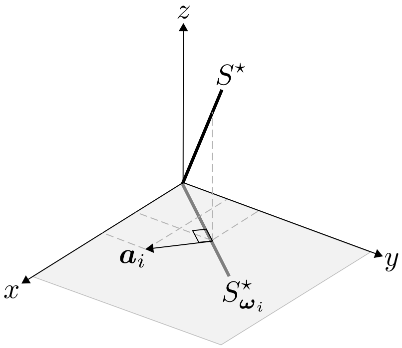

An entry in $\boldsymbol{a}_i$ is zero iff the corresponding entry in $\boldsymbol{\omega}_i$ is zero.

Sketch of the proof

One great thing:

- Every subspace in $\mathcal{S}(S^\star, \boldsymbol{\omega}_i)$ is orthogonal to $\boldsymbol{a}_i$.

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Cool! $\Rightarrow$

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$$\boldsymbol{A} = \left[\ \boldsymbol{a}_1 \ \middle|\ \cdots \ \middle|\ \boldsymbol{a}_N \ \right].$$

Sketch of the proof

One great thing:

- Every subspace in $\mathcal{S}(S^\star, \boldsymbol{\omega}_i)$ is orthogonal to $\boldsymbol{a}_i$.

Cool! $\Rightarrow$

- Construct

$$\boldsymbol{A} = \left[\ \boldsymbol{a}_1 \ \middle|\ \cdots \ \middle|\ \boldsymbol{a}_N\ \right].$$

- Every $S \in \mathcal{S}(S^\star, \boldsymbol{\Omega})$ must be contained in

$$\ker \boldsymbol{A}^\mathsf{T}.$$

Sketch of the proof

What we really want is to determine $\dim \ker \boldsymbol{A}^\mathsf{T}$.

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- If $\dim \ker \boldsymbol{A}^\mathsf{T} = r$
  $\Rightarrow$ Only $S^\star$ will agree with the projections. Moreover,

$$S^\star = \ker \boldsymbol{A}^\mathsf{T}$$

.

Sketch of the proof

- For any matrix $\boldsymbol{A}'$ formed with a subset of the columns in $\boldsymbol{A}$:

$$\boldsymbol{A}' = \underbrace{\begin{bmatrix} a_{11} & 0 & \\ a_{21} & a_{22} & \\ 0 & a_{32} & \\ 0 & 0 & \end{bmatrix}}_{n(\boldsymbol{A}') \,:=\, \#\text{columns}} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m(\boldsymbol{A}') := \#\text{nonzero rows}$$

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- We want $\dim\ker \boldsymbol{A}^{\mathsf{T}} = r$, so $\boldsymbol{A}$ better have $d - r$ linearly independent columns.

Sketch of the proof

We know how to deal with $\boldsymbol{A}$ using linear algebra!

- Through some technical details:

Lemma (Pimentel-Alarcón, Nowak, Boston, '14)

For almost every $S^\star$, the columns of $\boldsymbol{A}$ are linearly dependent if and only if $m(\boldsymbol{A}') < n(\boldsymbol{A}') + r$ for some matrix $\boldsymbol{A}'$ formed with a subset of the columns in $\boldsymbol{A}$.

Sketch of the proof

The zero entries of $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ are in the same positions.

$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longleftrightarrow \quad \boldsymbol{A}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \\ 0 & 0 & a_{43} \end{bmatrix}$$

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Then

$$m(\boldsymbol{\Omega}') \geq n(\boldsymbol{\Omega}') + r \iff m(\boldsymbol{A}') \geq n(\boldsymbol{A}') + r$$

□

Sketch of the proof

To wrap up:

- Every $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$,

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- Iff $S^{\star}$ is the only subspace in $\mathcal{S}(S^{\star}, \mathbf{\Omega})$. □

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Application

Low-Rank Matrix Completion (LRMC)

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- Given a subset of entries in a rank r matrix, exactly recover *all* of the missing entries.

$$\mathfrak{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathfrak{X}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

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- $\sim$ Identifying the subspace spanned by the columns, $S^\star$.

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- $\sim$ Identifying the subspace spanned by the columns, $S^\star$. Here

$$\hat{S} = \mathrm{span} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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- Maybe the real completion is:

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And the real subspace is

$$S^\star = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

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What if these assumptions are not met? How can we validate a completion?

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Corollary (Pimentel-Alarcón, Nowak, Boston, '14)

Let the columns of $\mathfrak{X}$ *be drawn independently according to* μ*, an absolutely continuous distribution with respect to the Lebesgue measure on* $S^\star$*. Suppose* $\mathfrak{X}_{\mathbf{\Omega}}$ *can be partitioned into two sets of columns,* $\mathfrak{X}_{\mathbf{\Omega}_1}$ *and* $\mathfrak{X}_{\mathbf{\Omega}_2}$*, such that* $\mathbf{\Omega}_2$ *satisfies the conditions of the subspace identifiability theorem.*
Let $\hat{S}$ *be the output of running an LRMC algorithm on* $\mathfrak{X}_{\mathbf{\Omega}_1}$*.*
Then for almost every $S^\star$*, and almost surely with respect to* μ*,* $\mathfrak{X}_{\mathbf{\Omega}_2}$ *fits in* $\hat{S}$ *if and only if* $\hat{S} = S^\star$*.*

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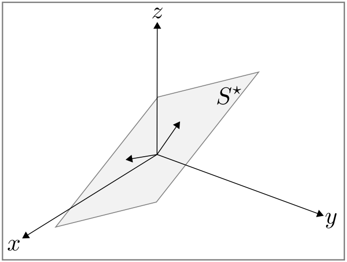

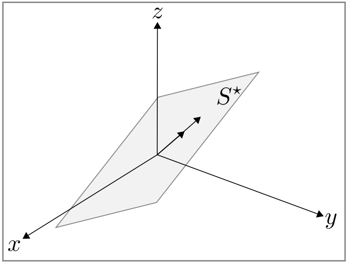

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Now we know that:

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- If and only if every subset of n columns of $\mathbf{\Omega}$ has at least $n + r$ nonzero rows.

Conclusions

Now we know that:

- It is possible to uniquely identify an r-dimensional subspace $S^\star$ from its projections onto $\mathbf{\Omega}$.
- If and only if every subset of n columns of $\mathbf{\Omega}$ has at least $n + r$ nonzero rows.
- Whence $S^\star = \ker \boldsymbol{A}^\mathsf{T}$.

Thanks.