A Characterization of Deterministic Sampling Patterns for Low-Rank Matrix Completion

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Outline

- Introduction
- When can we Low-Rank Matrix Complete?
- The Answer
- Implications
- Idea of the proof
- Conclusions
- Open questions (if time allows)
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Introduction

We have lots of data
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And we want to analyze it.
Linear Algebra is one of our favorite tools.
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- Because data is often well-modeled by linear subspaces.

\[
\begin{bmatrix}
1 & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 2 & 2 \\
2 & 4 & 2 & 6 & 4 & 2 & 6 & 2 & 4 & 4 \\
3 & 6 & 3 & 9 & 6 & 3 & 9 & 3 & 6 & 6 \\
1 & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 2 & 2 \\
2 & 4 & 2 & 6 & 4 & 2 & 6 & 2 & 4 & 4 \\
3 & 6 & 3 & 9 & 6 & 3 & 9 & 3 & 6 & 6 \\
\end{bmatrix}
\]
Introduction

Linear Algebra is one of our favorite tools.

- Because data is often well-modeled by linear subspaces.

We know how to find the subspace (e.g., using SVD).
Introduction

That’s all very nice, but... often data is missing!
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That’s all very nice, but... **often data is missing!**

- Example: Vision.

Image: Hopkins 155 Dataset
Introduction

That’s all very nice, but... often data is missing!

- Example: Vision.

![Image: Hopkins 155 Dataset]

- We still want to find subspaces.
Low-Rank Matrix Completion (LRMC) aims to find the subspace from incomplete datasets.
Introduction

Low-Rank Matrix Completion:
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Low-Rank Matrix Completion:

- Given a subset of entries in a rank-$r$ matrix, exactly recover all of the missing entries.
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Low-Rank Matrix Completion:

- Given a subset of entries in a rank-\( r \) matrix, exactly recover all of the missing entries.

\[
\begin{bmatrix}
1 & \cdot & 3 & \cdot \\
1 & 2 & \cdot & \cdot \\
\cdot & 2 & 3 & \cdot \\
\cdot & \cdot & \cdot & 4 \\
\cdot & \cdot & \cdot & 4
\end{bmatrix}
\]
Low-Rank Matrix Completion:

- Given a subset of entries in a rank-$r$ matrix, exactly recover all of the missing entries.

$$X_{\Omega} = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix} \quad \Rightarrow \quad X = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$
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- Identifying the subspace spanned by the columns, \( S^* \).
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Low-Rank Matrix Completion:

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- Identifying the subspace spanned by the columns, $S^*$. Here

\[
S^* = \text{span} \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}.
\]
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When can we Low-Rank Matrix Complete?

- Existing theory (e.g. Candès and Recht, ’09) essentially says:

  - If entries are observed uniformly at random.
  - If matrix is incoherent, but not necessarily a sufficient condition.
  - Generally unverifiable or unjustified in practice.
  - Then with high probability you can complete the matrix.

But what if these assumptions are not met?
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- Existing theory (e.g. Candès and Recht, ’09) essentially says:
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But what if these assumptions are not met?
When can we Low-Rank Matrix Complete?

- What makes a matrix *completable*?
When can we Low-Rank Matrix Complete?

- What makes a matrix complete?
- What conditions must a matrix satisfy?
Outline

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The Answer

Setup

- $\Omega$ will indicate the observed entries:
The Answer

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$$X_{\Omega} = \begin{bmatrix}
1 & \cdot & 3 & \cdot \\
1 & 2 & \cdot & \cdot \\
\cdot & 2 & 3 & \cdot \\
\cdot & \cdot & \cdot & 4 \\
\cdot & \cdot & \cdot & 4
\end{bmatrix}$$
Setup

- $\Omega$ will indicate the observed entries:

\[
\begin{align*}
X_\Omega &= \begin{bmatrix}
1 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 4 \\
\end{bmatrix} \\
\Omega &= \begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\end{align*}
\]
The Answer

Setup

- For any matrix $\Omega'$ formed with a subset of the columns in $\Omega$:

$$\Omega' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} m(\Omega') := \#\text{nonzero rows} \\ n(\Omega') := \#\text{columns} \end{array} \right.$$

[Matrix and equations]
The Answer

Theorem (P.-A., Nowak, Boston (Allerton ’15))

For almost every $X$, there exist at most finitely many rank-$r$ completions of $X_{\Omega}$ if and only if every matrix $\Omega'$ formed with a subset of the columns in $\Omega$ satisfies

$$m(\Omega') \geq n(\Omega')/r + r.$$
The Answer

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There is a set of measure zero of bad matrices for which this theorem does not apply.
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$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
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1 & 0 & 0
\end{bmatrix}$$

$$X_\Omega = \begin{bmatrix}
0 & \cdot & \cdot \\
0 & 0 & \cdot \\
\cdot & 0 & 0 \\
\cdot & \cdot & 0
\end{bmatrix}$$
For almost every $X$, there exist at most finitely many rank-$r$ completions of $X_\Omega$ if and only if every matrix $\Omega'$ formed with a subset of the columns in $\Omega$ satisfies

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The Answer

For almost every \( \mathbf{X} \), there exist at most finitely many rank-\( r \) completions of \( \mathbf{X}_\Omega \) if and only if every matrix \( \Omega' \) formed with a subset of the columns in \( \Omega \) satisfies

\[
m(\Omega') \geq n(\Omega')/r + r.
\]

\[
\mathbf{X}_\Omega = \begin{bmatrix}
1 & 1 & 3 & \cdot \\
1 & 2 & \cdot & 1 \\
3 & \cdot & 5 & 4 \\
\cdot & 7 & 6 & 5
\end{bmatrix}
\]
The Answer

For almost every $\mathbf{X}$, there exist at most finitely many rank-$r$ completions of $\mathbf{X}_\Omega$ if and only if every matrix $\Omega'$ formed with a subset of the columns in $\Omega$ satisfies

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This is the answer!
The Answer

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This is the answer!

Every subset of $n$ columns of $\Omega$ has at least $n/r + r$ nonzero rows.
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$$\Omega = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
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$$\Omega = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \text{Check:} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}.$$
The answer

Now we know when there are at most \textit{finitely} many completions.

• Then what?
Theorem (P.-A., Nowak, Boston (Allerton ’15))

If in addition $X_{\Omega}$ has an extra $(d - r)$ columns observed on $\hat{\Omega}$, such that every matrix $\Omega'$ formed with a subset of the columns in $\hat{\Omega}$ satisfies

$$m(\Omega') \geq n(\Omega') + r,$$

then $X$ can be uniquely recovered from $X_{\Omega}$. 
The Answer (in words)

If a matrix does not satisfy our sampling conditions, then you *cannot* complete it.
If a matrix does not satisfy our sampling conditions, then you **cannot** complete it.

\[X_{\Omega} = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix}\]
The Answer (in words)

If a matrix satisfies our sampling conditions, then you can finitely complete it.
If a matrix satisfies our sampling conditions, then you can finitely complete it.

\[ X_{\Omega} = \begin{bmatrix} 1 & 1 & 3 & \cdot \\ 1 & 2 & \cdot & 1 \\ 3 & \cdot & 5 & 4 \\ \cdot & 7 & 6 & 5 \end{bmatrix} \]
If a matrix satisfies our sampling conditions, then you can **finitely** complete it.

\[
X_\Omega = \begin{bmatrix}
1 & 1 & 3 & \cdot \\
1 & 2 & \cdot & 1 \\
3 & \cdot & 5 & 4 \\
\cdot & 7 & 6 & 5
\end{bmatrix}
\]

Sometimes **finitely** completable = **uniquely** completable (e.g., rank= 1), but sometimes not.
The Answer (in words)

But just a few additional samples in a *finitely* completable matrix make it *uniquely* completable.
But just a few additional samples in a finitely completable matrix make it uniquely completable.

\[ X_\Omega = \begin{bmatrix} 1 & 1 & 3 & \cdot & -1 & 1 \\ 1 & 2 & \cdot & 1 & \cdot & -1 \\ 3 & \cdot & 5 & 4 & 3 & \cdot \\ \cdot & 7 & 6 & 5 & 5 & -2 \end{bmatrix} \]
The Answer (take home message)

In essence:

$r$ complete columns (linearly independent) uniquely define an $r$-dimensional subspace.
In essence:

**r complete** columns (linearly independent) uniquely define an $r$-dimensional subspace.

$(r + 1)(d - r)$ **incomplete** columns (observed in the right places) uniquely define an $r$-dimensional subspace.
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- Implications
- Idea of the proof
- Conclusions
Implications (coherence)

  - For almost every matrix, $\Theta(\max\{r, \log d\})$ uniform random entries per column are sufficient for completion.
Implications (coherence)

  - For almost every matrix, $\Theta(\max\{r, \log d\})$ uniform random entries per column are sufficient for completion.
- Regardless of coherence!
Implications

- Our results tell us exactly which entries to observe.
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  - We can now design Adaptive LRMC Algorithms.
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- Help answer an important open question:
Implications

- Our results tell us exactly which entries to observe.
  - We can now design Adaptive LRMC Algorithms.
- Help answer an important open question:
  - The Sample Complexity of Subspace Clustering with Missing Data.
Implications

Validation criteria:

Suppose you observe the right entries.

Try to complete the matrix using any method.

If you find a rank-$r$ completion, then it is the right completion.

In lieu of coherence assumptions.

In lieu of uniform sampling assumptions.

With probability 1 (as opposed to with high probability).
Implications

Validation criteria:
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Validation criteria:

- Suppose you observe **the right entries**.
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- In lieu of **coherence** assumptions.
- In lieu of **uniform sampling** assumptions.
Implications

Validation criteria:

- Suppose you observe the right entries.
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- If you find a rank-$r$ completion, then it is the right completion.
- In lieu of coherence assumptions.
- In lieu of uniform sampling assumptions.
- With probability 1 (as opposed to with high probability).
Implications: better understanding of sampling regimes
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Previously known under random samplings

\[ \Theta(r \log d) \]

\[ N \]

Samples per column

Columns
Implications: better understanding of sampling regimes

\[ \ell = \frac{r(d - r)}{N} + r \]

Previously known under random samplings

Impossible
Implications: better understanding of sampling regimes

\[ \ell = \frac{r(d - r)}{N} + r \]

Previously known under random samplings

Possible if entries are observed in at the right places
Implications: better understanding of sampling regimes

\[ \ell = \frac{r(d-r)}{N} + r \]

Previously known under random samplings

Possible if entries are observed in at the right places

\[ \Theta(r \log d) \]
\[ \Theta(\max\{r, \log d\}) \]
\[ r + 1 \]

Columns

Possible under random samplings

Samples per column
A column with $r + 1$ samples imposes one restriction on what the subspace may be.
Idea of the proof

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$$X_{\Omega} = \begin{bmatrix} x_{\omega_1} \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
Idea of the proof

A column with $r + 1$ samples imposes one restriction on what the subspace may be.

$$X = \begin{bmatrix} x_{\omega_1} \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This reduces one degree of freedom in the Grassmannian.
A column with \( r + 1 \) samples imposes one restriction on what the subspace may be.

\[
X = \begin{bmatrix} x_{\omega_1} \\ 1 \\ 1 \\ 1 \end{bmatrix}
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- A subspace \( S \) fits \( x_{\omega_1} \) if and only if \( f_1(S) = 0 \).
Idea of the proof

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. 
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- A subspace \( S \) fits \( x_{\omega_1} \) \( \iff \) \( f_1(S) = 0 \).
- This reduces one degree of freedom in the Grassmannian.
Idea of the proof

An other column with $r+1$ samples imposes an other restriction.

$$X_{\Omega} = \begin{bmatrix} x_{\omega_2} \\ \cdot \\ 2 \\ 2 \end{bmatrix}$$
Idea of the proof

An other column with $r + 1$ samples imposes an other restriction.

$$X_{\Omega} = \begin{bmatrix} x_{\omega_2} \\ \cdot \\ 2 \\ 2 \end{bmatrix}$$
Idea of the proof

An other column with \( r + 1 \) samples imposes an other restriction.

\[
X_{\Omega} = \begin{bmatrix}
    x_{\omega_2} \\
    \vdots \\
    2 \\
    2
\end{bmatrix}
\]

- A subspace \( S \) fits \( x_{\omega_2} \) \( \iff \) \( f_2(S) = 0 \).
Idea of the proof

Each column with $r + 1$ samples imposes one restriction.
Idea of the proof

Each column with $r + 1$ samples imposes one restriction.

$$X_\Omega = \begin{bmatrix} x_{\omega_1} & x_{\omega_2} \\ 1 & \cdot \\ 1 & 2 \\ \cdot & 2 \end{bmatrix}$$
Idea of the proof

Each column with \( r + 1 \) samples imposes one restriction.

\[
X_{\Omega} = \begin{bmatrix}
    x_{\omega_1} & x_{\omega_2} \\
    1 & . \\
    1 & 2 \\
    . & 2
\end{bmatrix}
\]
Idea of the proof

Each column with $r + 1$ samples imposes one restriction.

$$X_\Omega = \begin{bmatrix} x_{\omega_1} & x_{\omega_2} \\ 1 & . \\ 1 & 2 \\ . & 2 \end{bmatrix}$$

- A subspace $S$ fits $X_\Omega \iff \begin{cases} f_1(S) = 0 \\ f_2(S) = 0 \end{cases}$.
Idea of the proof

- Each column with $r + 1$ samples imposes one restriction:

$$f_1, f_2, \ldots, f_N.$$
Idea of the proof

- Each column with \( r + 1 \) samples imposes one restriction: 
  
  \[
  f_1, f_2, \ldots, f_N.
  \]

- The Grassmannian has \( r(d - r) \) degrees of freedom.
Idea of the proof

- Each column with \( r + 1 \) samples imposes one restriction:
  \[
  f_1, f_2, \ldots, f_N.
  \]
- The Grassmannian has \( r(d - r) \) degrees of freedom.
- If we have \( r(d - r) \) not redundant restrictions:
Idea of the proof

- Each column with $r + 1$ samples imposes one restriction:
  \[ f_1, f_2, \ldots, f_N. \]

- The Grassmannian has $r(d - r)$ degrees of freedom.
- If we have $r(d - r)$ not redundant restrictions:
  - We can identify $S^*$ up to finite choice.
Idea of the proof

- \( f_i(S) \) only involves the variables (of \( S \)) corresponding to the nonzero rows of \( \omega_i \).
Idea of the proof

- $f_i(S)$ only involves the variables (of $S$) corresponding to the nonzero rows of $\omega_i$.
- We want all sets of $n$ polynomials to involve at least $n/r + r$ variables (otherwise they will be dependent).
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Conclusions

- In essence:
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Our results tell us when will a set of incomplete vectors uniquely define an $r$-dimensional subspace (just as a set of $r$ linearly independent complete vectors would).
Conclusions

- In essence:

  Our results tell us when will a set of incomplete vectors uniquely define an \( r \)-dimensional subspace (just as a set of \( r \) linearly independent complete vectors would).

- This sheds new light on LRMC.
Thanks.
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Open questions

It is one thing to be *theoretically* able to complete a matrix; an other one to complete it *efficiently*.
Open questions

It is one thing to be theoretically able to complete a matrix; an other one to complete it efficiently.

Open questions

It is one thing to be *theoretically* able to complete a matrix; another one to complete it *efficiently*.

  - New sampling regimes where you can *theoretically* complete a matrix.
Open questions

It is one thing to be theoretically able to complete a matrix; another one to complete it efficiently.

P.-A., Nowak, Boston (Allerton ’15):
- New sampling regimes where you can theoretically complete a matrix.
- This may involve solving a complex system of polynomial equations!
Open questions

It is one thing to be theoretically able to complete a matrix; another one to complete it efficiently.

  - New sampling regimes where you can theoretically complete a matrix.
  - This may involve solving a complex system of polynomial equations!
  - This is computationally prohibitive.
Open questions

Does **missingness** come at a price?
Open questions

Does **missingness** come at a price?
Open questions

Does **missingness** come at a price?

How much **missing data** can we handle and remain **computationally efficient**?
Open questions

Can practical algorithms complete coherent matrices?
Open questions

Can practical algorithms complete coherent matrices?
Open questions

Does coherence come at a price?
Open questions

Does coherence come at a price?
Open questions

Does coherence come at a price?

How much coherence can we handle and remain computationally efficient?
Thanks again!
Thanks again!
(this time I’m really done)