A Characterization of Deterministic Sampling Patterns for Low-Rank Matrix Completion

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Allerton, 2015

Outline

- Introduction
- When can we Low-Rank Matrix Complete?
- The Answer
- Implications
- Idea of the proof
- Conclusions
- Open questions (if time allows)

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And we want to analyze it.

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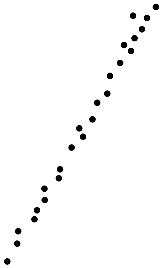

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 2 & 2 \\ 2 & 4 & 2 & 6 & 4 & 2 & 6 & 2 & 4 & 4 \\ 3 & 6 & 3 & 9 & 6 & 3 & 9 & 3 & 6 & 6 \\ 1 & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 2 & 2 \\ 2 & 4 & 2 & 6 & 4 & 2 & 6 & 2 & 4 & 4 \\ 3 & 6 & 3 & 9 & 6 & 3 & 9 & 3 & 6 & 6 \end{bmatrix}$$

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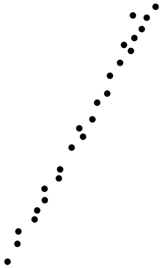

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- We know how to find the subspace (e.g., using SVD).

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- We still want to find subspaces.

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Low-Rank Matrix Completion (LRMC) aims to find the subspace from incomplete datasets.

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$$S^\star = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

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  But what if these assumptions are not met?

When can we Low-Rank Matrix Complete?

- What makes a matrix *completable*?

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- What conditions must a matrix satisfy?

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The Answer

Setup

- For any matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$:

$$\mathbf{\Omega}' = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{n(\mathbf{\Omega}') \,:=\, \#\text{columns}} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m(\mathbf{\Omega}') := \#\text{nonzero rows}$$

The Answer

Theorem (P.-A., Nowak, Boston (Allerton '15))

For almost every $\mathbf{X}$, there exist at most finitely many rank-r completions of $\mathbf{X}_{\mathbf{\Omega}}$ if and only if every matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$ satisfies

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$$\mathbf{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & 1 & 3 & \cdot \\ 1 & 2 & \cdot & 1 \\ 3 & \cdot & 5 & 4 \\ \cdot & 7 & 6 & 5 \end{bmatrix}$$

The Answer

For almost every $\mathbf{X}$, there exist at most finitely many rank-r completions of $\mathbf{X_\Omega}$ if and only if every matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$ satisfies

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$$\mathbf{X_\Omega} = \begin{bmatrix} 1 & 1 & 3 & \cdot \\ 1 & 2 & \cdot & 1 \\ 3 & \cdot & 5 & 4 \\ \cdot & 7 & 6 & 5 \end{bmatrix} \qquad \Rightarrow \begin{cases} \mathbf{X} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 2 & 1 & 1 \\ 3 & 5 & 5 & 4 \\ 4 & 7 & 6 & 5 \end{bmatrix} \\ \\ \mathbf{X} = \begin{bmatrix} 1 & 1 & 3 & 10 \\ 1 & 2 & \frac{21}{13} & 1 \\ 3 & \frac{53}{9} & 5 & 4 \\ \frac{68}{19} & 7 & 6 & 5 \end{bmatrix} \end{cases}$$

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For almost every $\mathbf{X}$, there exist at most finitely many rank-r completions of $\mathbf{X}_{\mathbf{\Omega}}$ if and only if every matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$ satisfies

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$$\mathbf{\Omega} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \text{ Check: } \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The answer

Now we know when there are at most finitely many completions.

- Then what?

The Answer

Theorem (P.-A., Nowak, Boston (Allerton '15))

If in addition $\mathbf{X}_{\mathbf{\Omega}}$ has an extra $(d-r)$ columns observed on $\hat{\mathbf{\Omega}}$, such that every matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\hat{\mathbf{\Omega}}$ satisfies

$$m(\mathbf{\Omega}') \;\geq\; n(\mathbf{\Omega}') + r,$$

then $\mathbf{X}$ can be uniquely recovered from $\mathbf{X}_{\mathbf{\Omega}}$.

The Answer (in words)

If a matrix does not satisfy our sampling conditions, then you cannot complete it.

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$$\mathbf{X}_{\boldsymbol{\Omega}} = \begin{bmatrix} 1 & 1 & 3 & \cdot \\ 1 & 2 & \cdot & 1 \\ 3 & \cdot & 5 & 4 \\ \cdot & 7 & 6 & 5 \end{bmatrix}$$

Sometimes finitely completable = uniquely completable (e.g., rank$= 1$), but sometimes not.

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$$\mathbf{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & 1 & 3 & \cdot & -1 & 1 \\ 1 & 2 & \cdot & 1 & \cdot & -1 \\ 3 & \cdot & 5 & 4 & 3 & \cdot \\ \cdot & 7 & 6 & 5 & 5 & -2 \end{bmatrix}$$

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In essence:

r **complete** columns (linearly independent) uniquely define an r-dimensional subspace.

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$(r+1)(d-r)$ **incomplete** columns (observed in the right places) uniquely define an r-dimensional subspace.

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- P.-A., Nowak, Boston (Allerton '15):
  - For almost every matrix, $\mathcal{O}(\max\{r, \log d\})$ uniform random entries per column are sufficient for completion.
- Regardless of coherence!

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- Help answer an important open question:
  - The Sample Complexity of Subspace Clustering with Missing Data.

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- In lieu of uniform sampling assumptions.
- With probability 1 (as opposed to *with high probability*).

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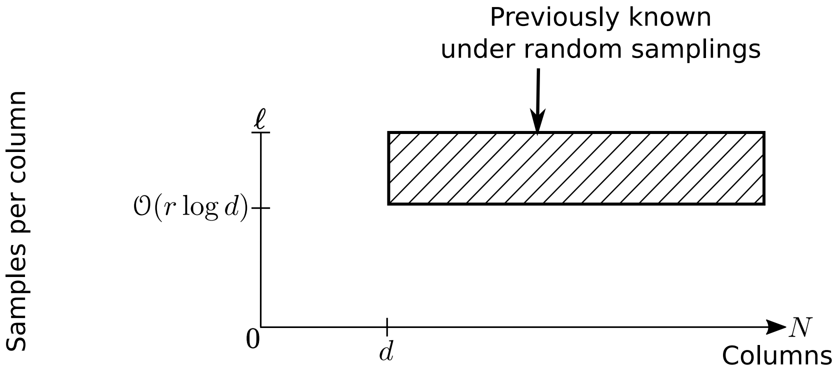

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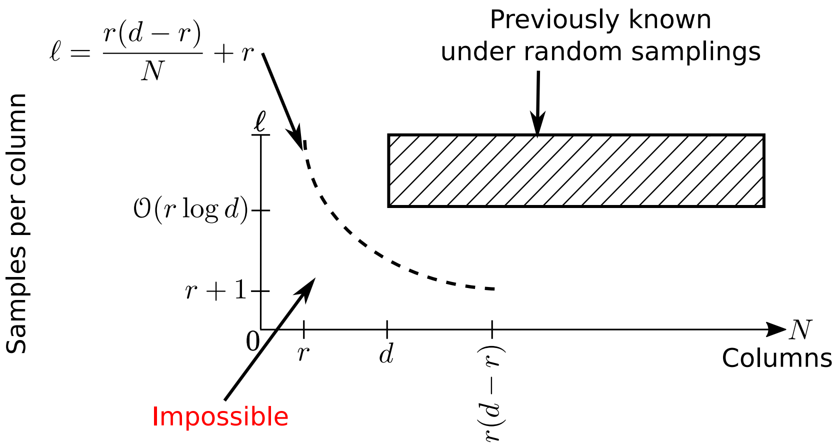

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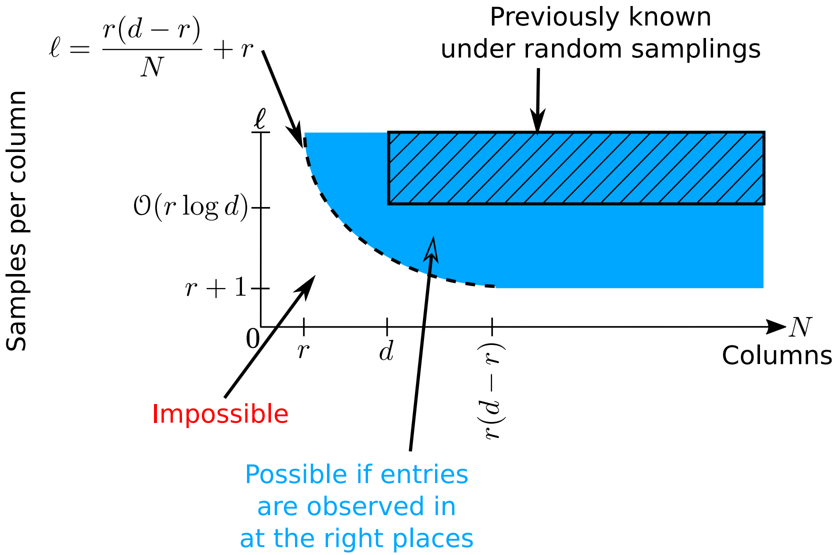

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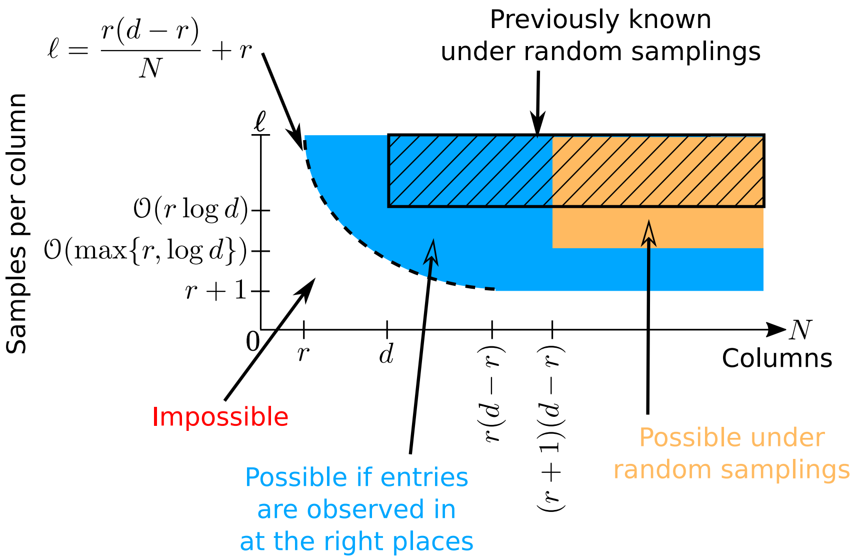

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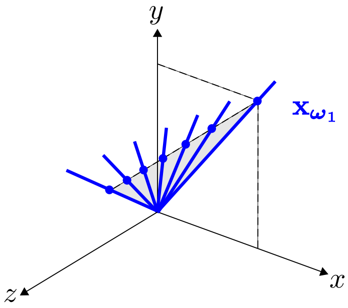

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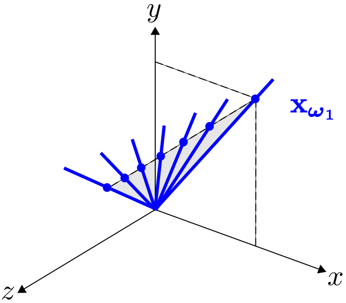

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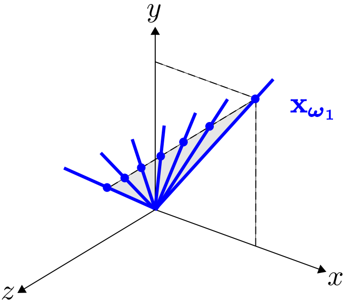

- A subspace S *fits* $\mathbf{x}_{\boldsymbol{\omega}_1} \iff f_1(S) = 0$.
- This reduces one degree of freedom in the Grassmannian.

Idea of the proof

An other column with $r+1$ samples imposes an other restriction.

$$\mathbf{X}_{\boldsymbol{\Omega}} \quad = \quad \begin{array}{c} \mathbf{x}_{\boldsymbol{\omega}_2} \\ \left[\begin{array}{c} \cdot \\ 2 \\ 2 \end{array}\right] \end{array}$$

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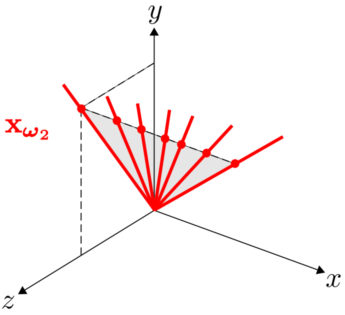

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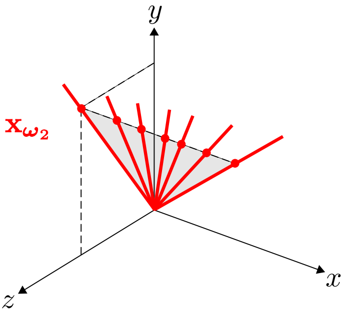

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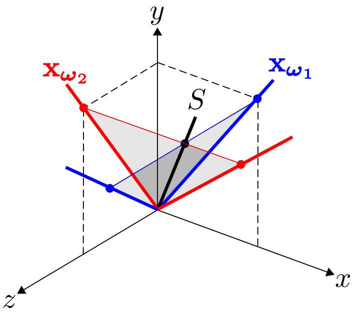

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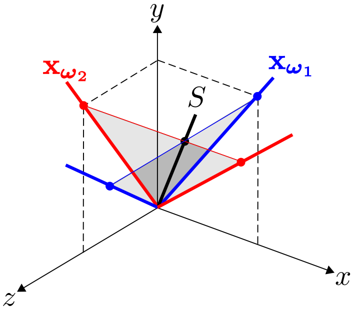
\begin{array}{cc}
 & \begin{array}{cc} \mathbf{x}_{\omega_1} & \mathbf{x}_{\omega_2} \end{array} \\
\mathbf{X}_{\Omega} \;=\; & \left[\begin{array}{cc} 1 & \cdot \\ 1 & 2 \\ \cdot & 2 \end{array}\right]
\end{array}
$$



- A subspace S *fits* $\mathbf{X}_{\Omega}$ $\iff \left\{ \begin{array}{l} f_1(S) = 0 \\ f_2(S) = 0 \end{array} \right.$.

Idea of the proof

- Each column with $r + 1$ samples imposes one restriction:

$$f_1, f_2, \ldots, f_N.$$

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- We want all sets of n polynomials to involve at least $n/r + r$ variables (otherwise they will be dependent)

Outline

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- When can we Low-Rank Matrix Complete? ✓
- The Answer ✓
- Implications ✓
- Idea of the proof ✓
- Conclusions

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Our results tell us when will a set of incomplete vectors uniquely define an r-dimensional subspace (just as a set of r linearly independent complete vectors would).

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- This sheds new light on LRMC.

Thanks.

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It is one thing to be theoretically able to complete a matrix; an other one to complete it efficiently.

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- P.-A., Nowak, Boston (Allerton '15):
  - New sampling regimes where you can theoretically complete a matrix.
  - This may involve solving a complex system of polynomial equations!
  - This is computationally prohibitive.

Open questions

Does missingness come at a price?

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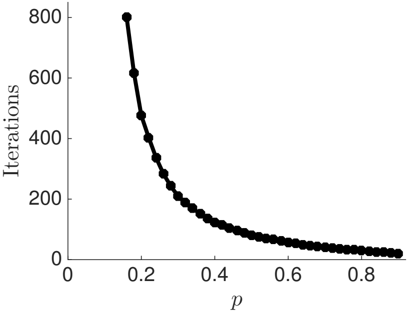

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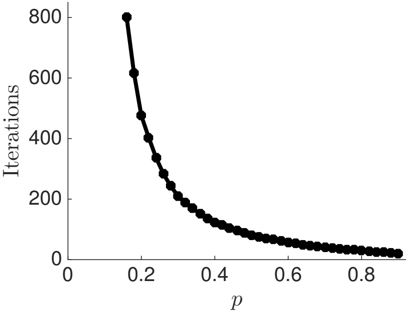

How much missing data can we handle and remain computationally efficient?

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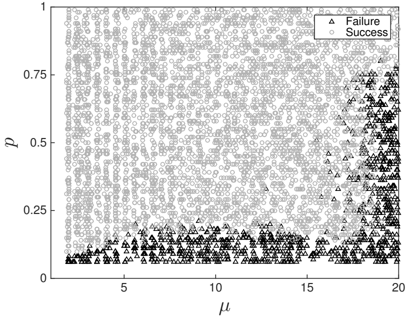

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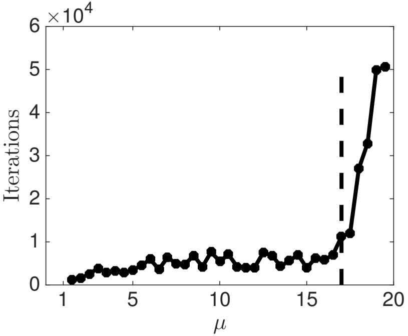

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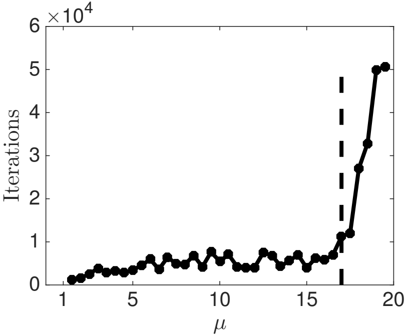

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(this time I'm really done)