Learning Subspaces by Pieces

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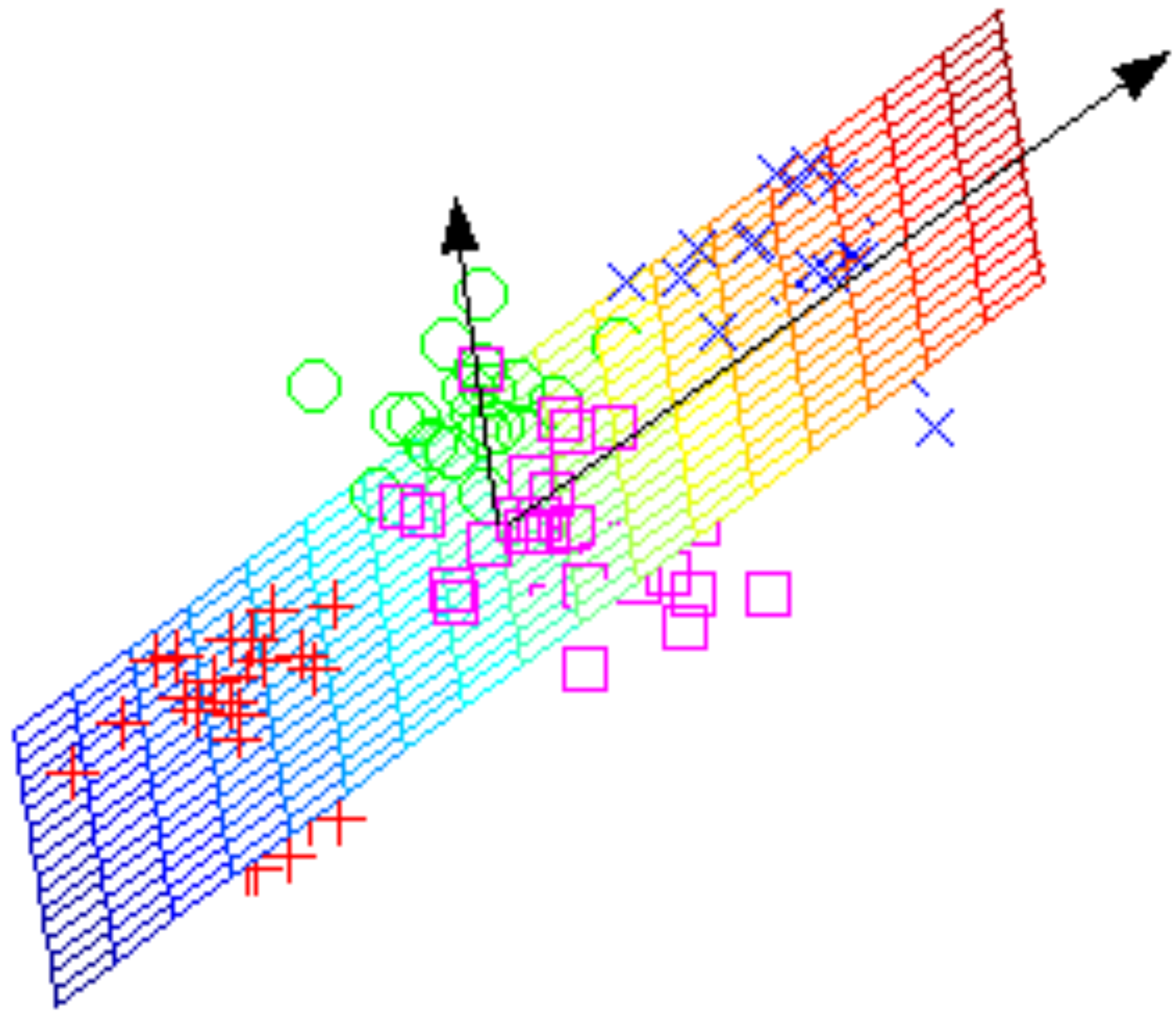
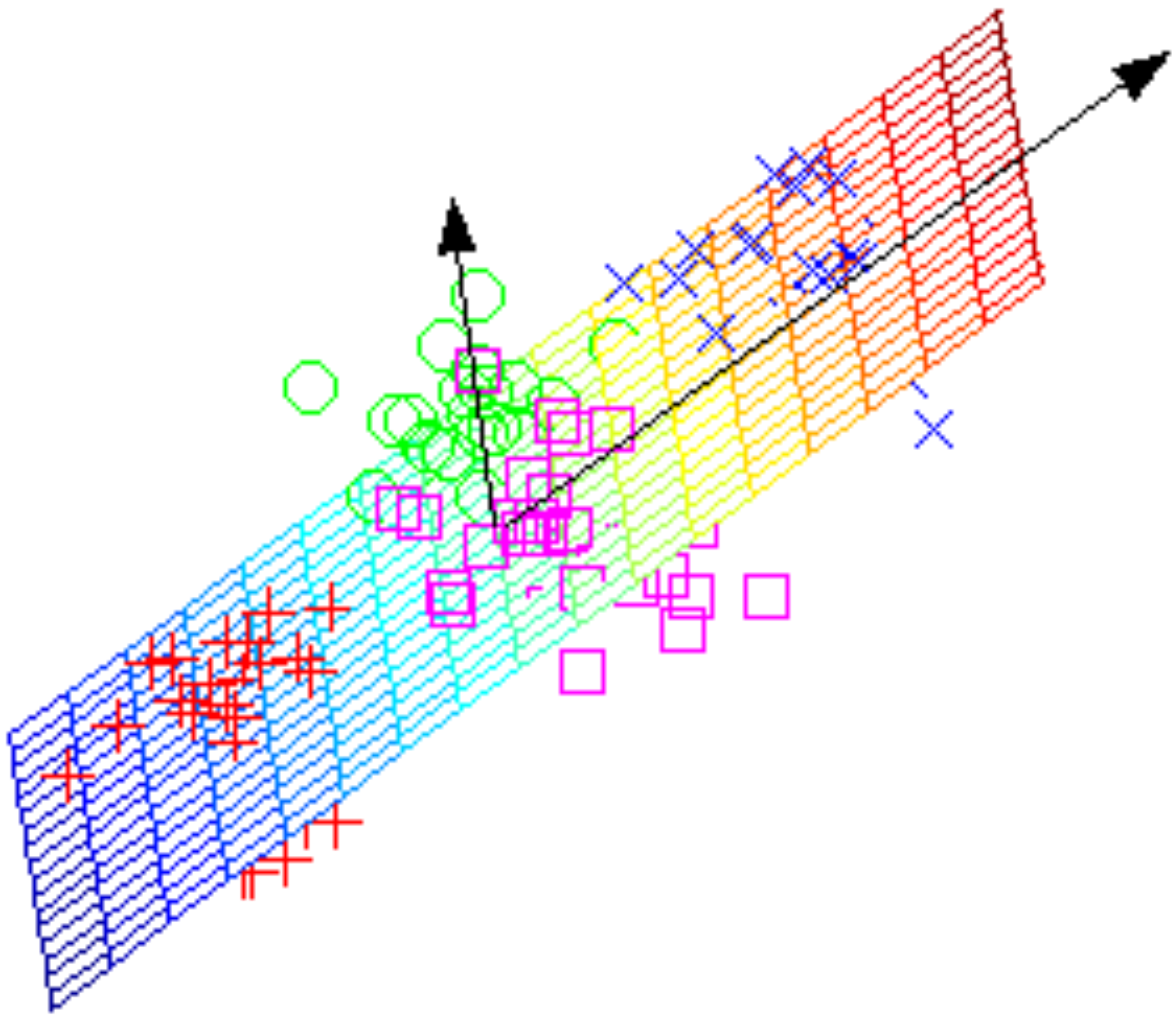
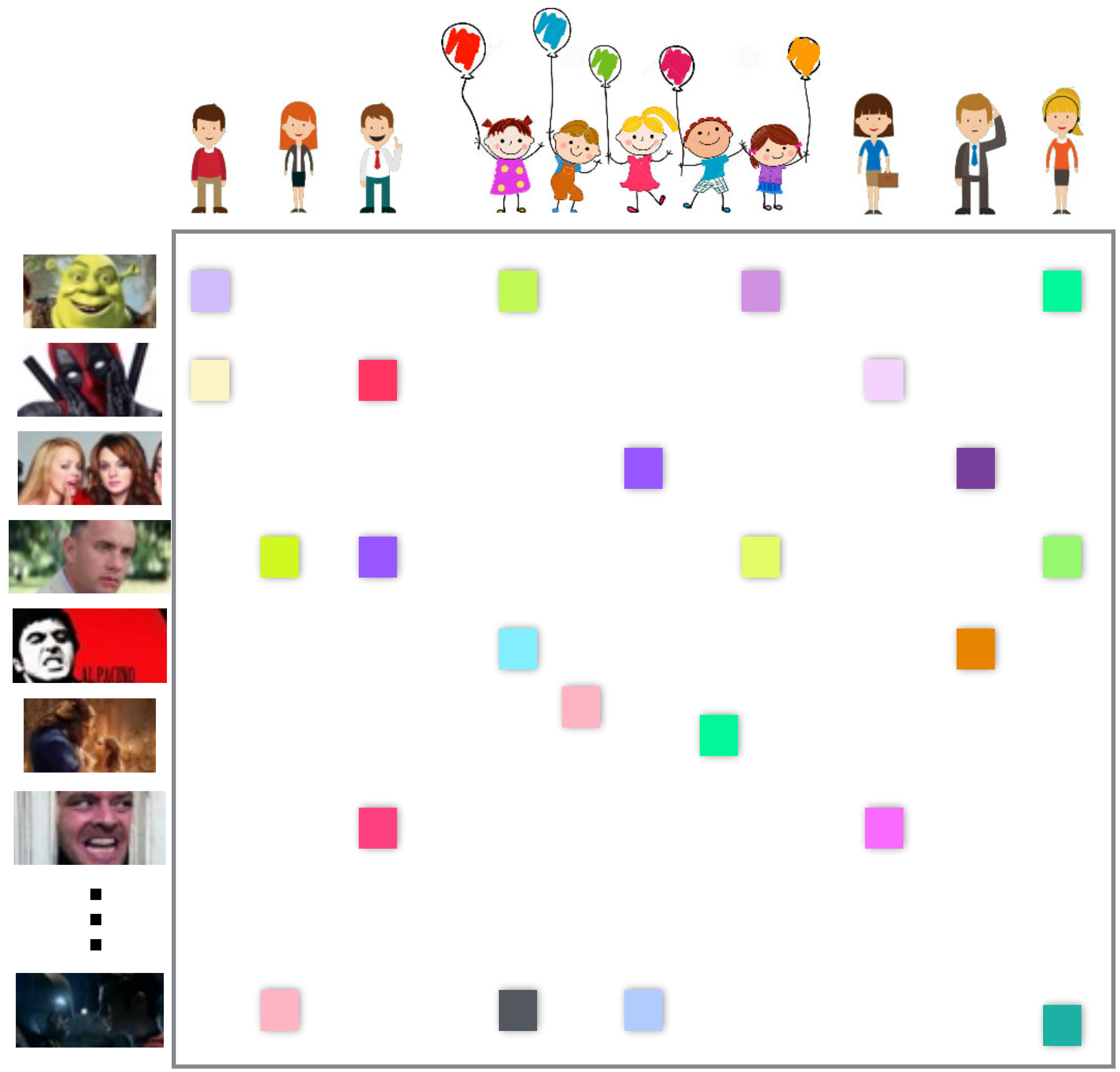
Department of Electrical and Computer Engineering



Subspaces in Big Data

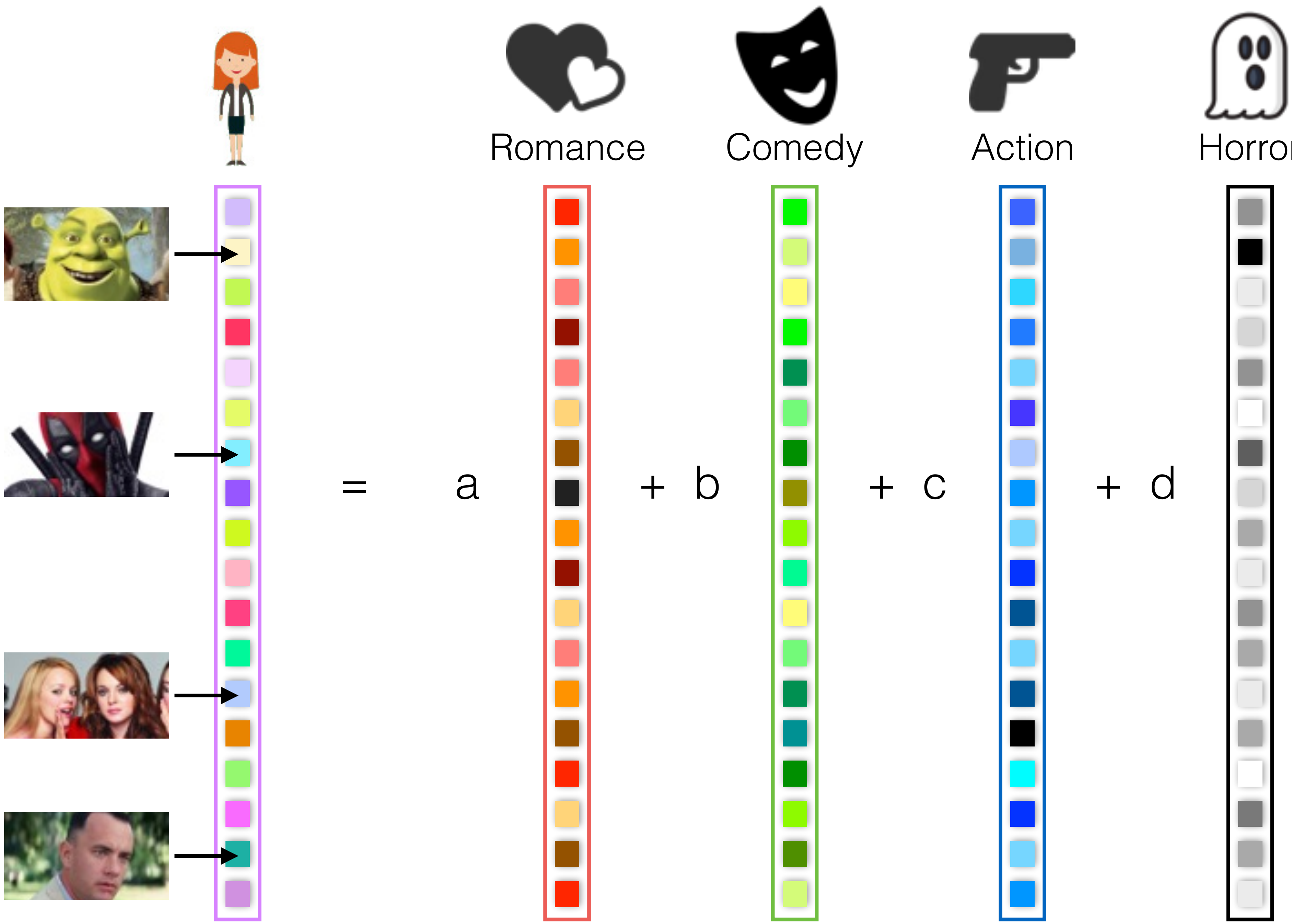


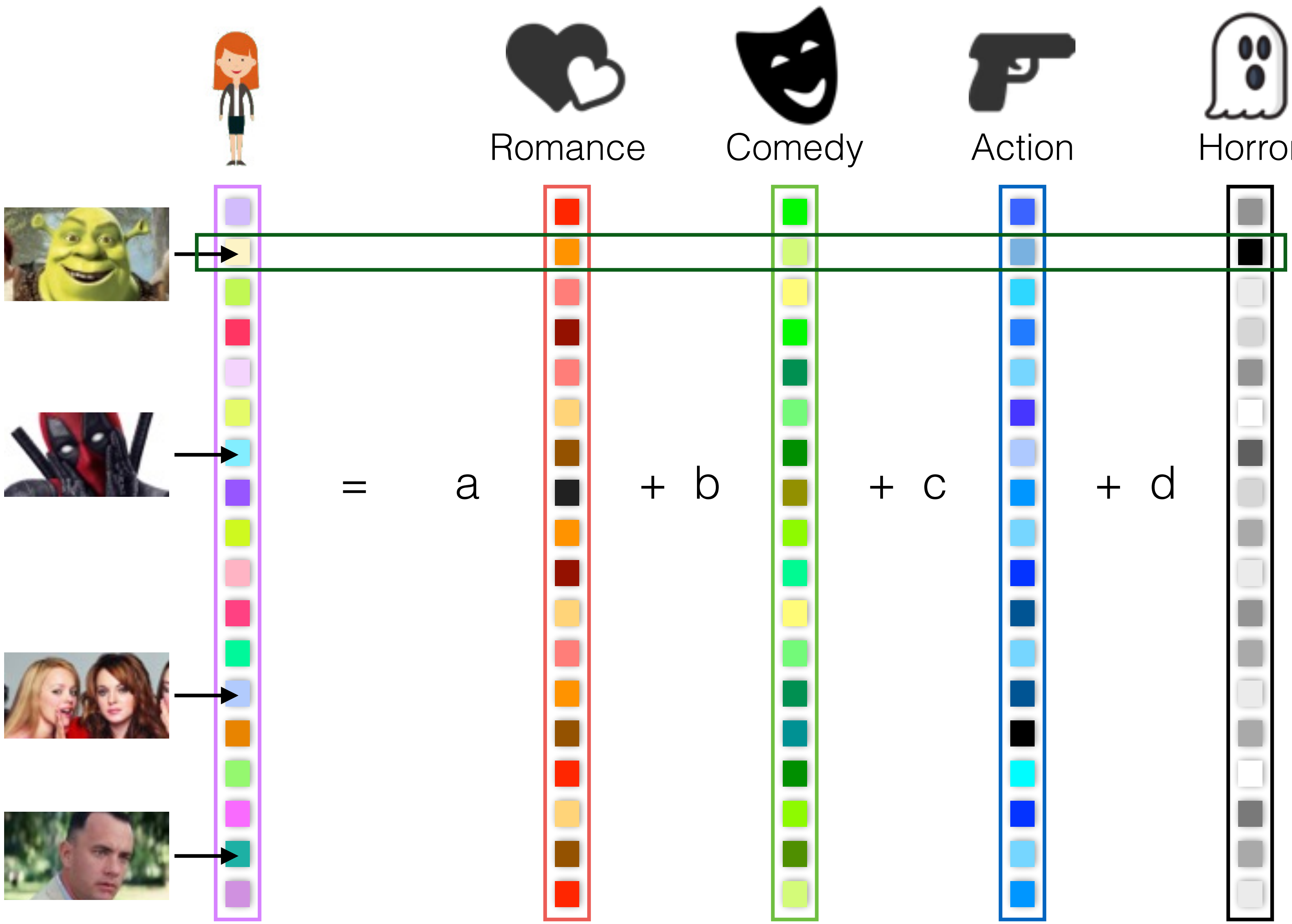
Subspaces in Big (incomplete) Data



Textbook Example

Recommender Systems


Romance
Comedy
Action
Horror
= a
+ b
+ c
+ d



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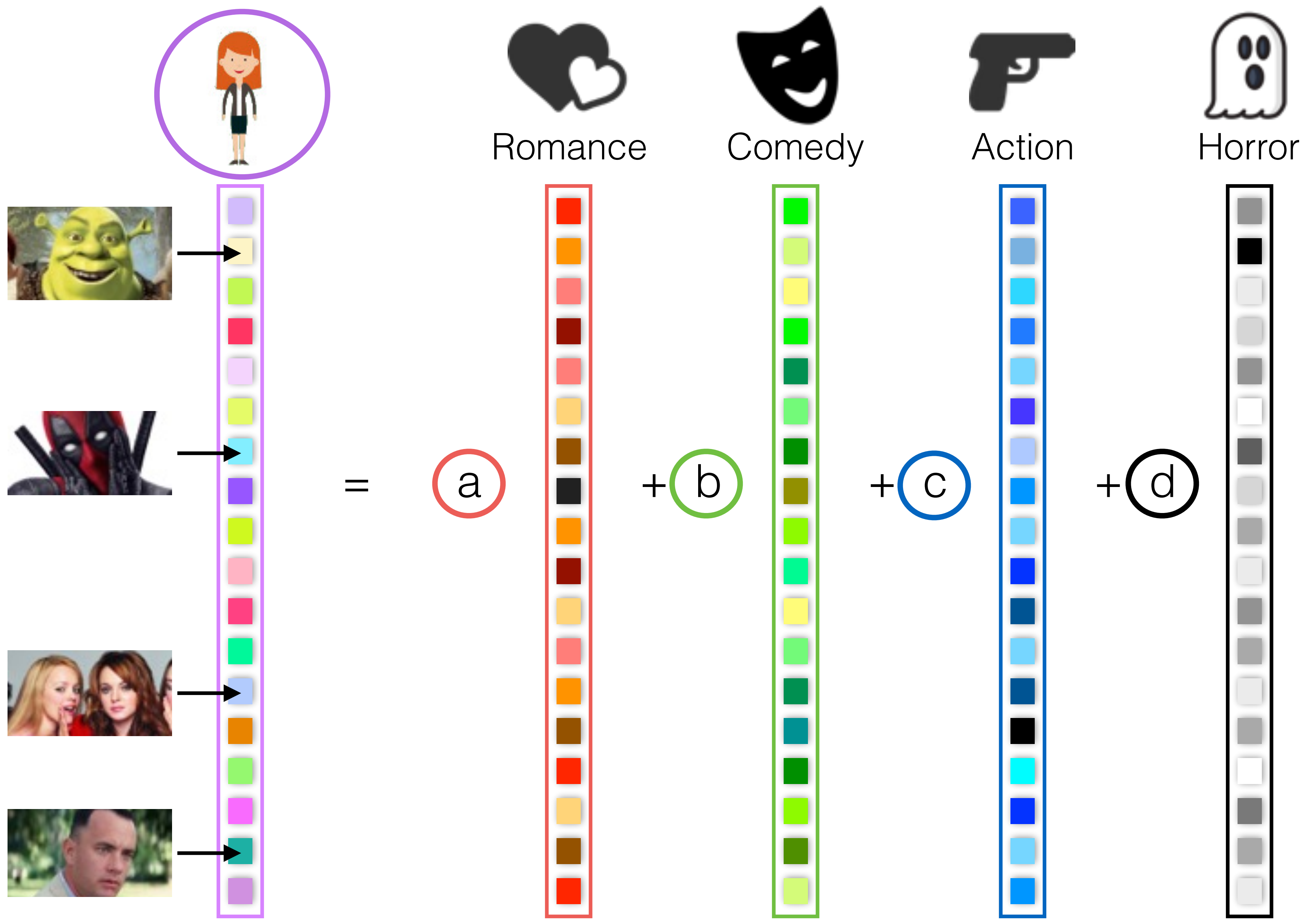

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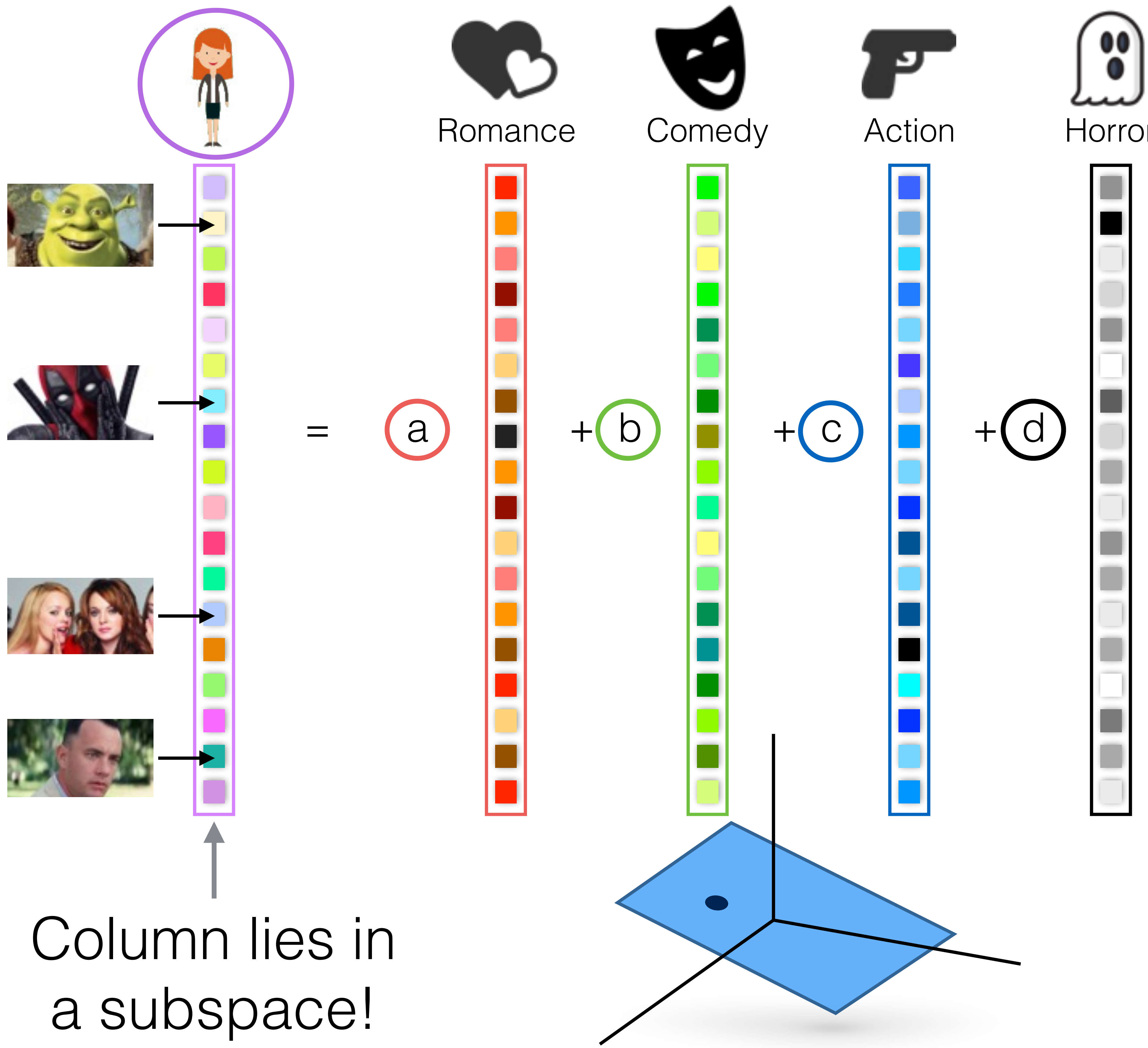

Romance
Comedy
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Horror
=
a
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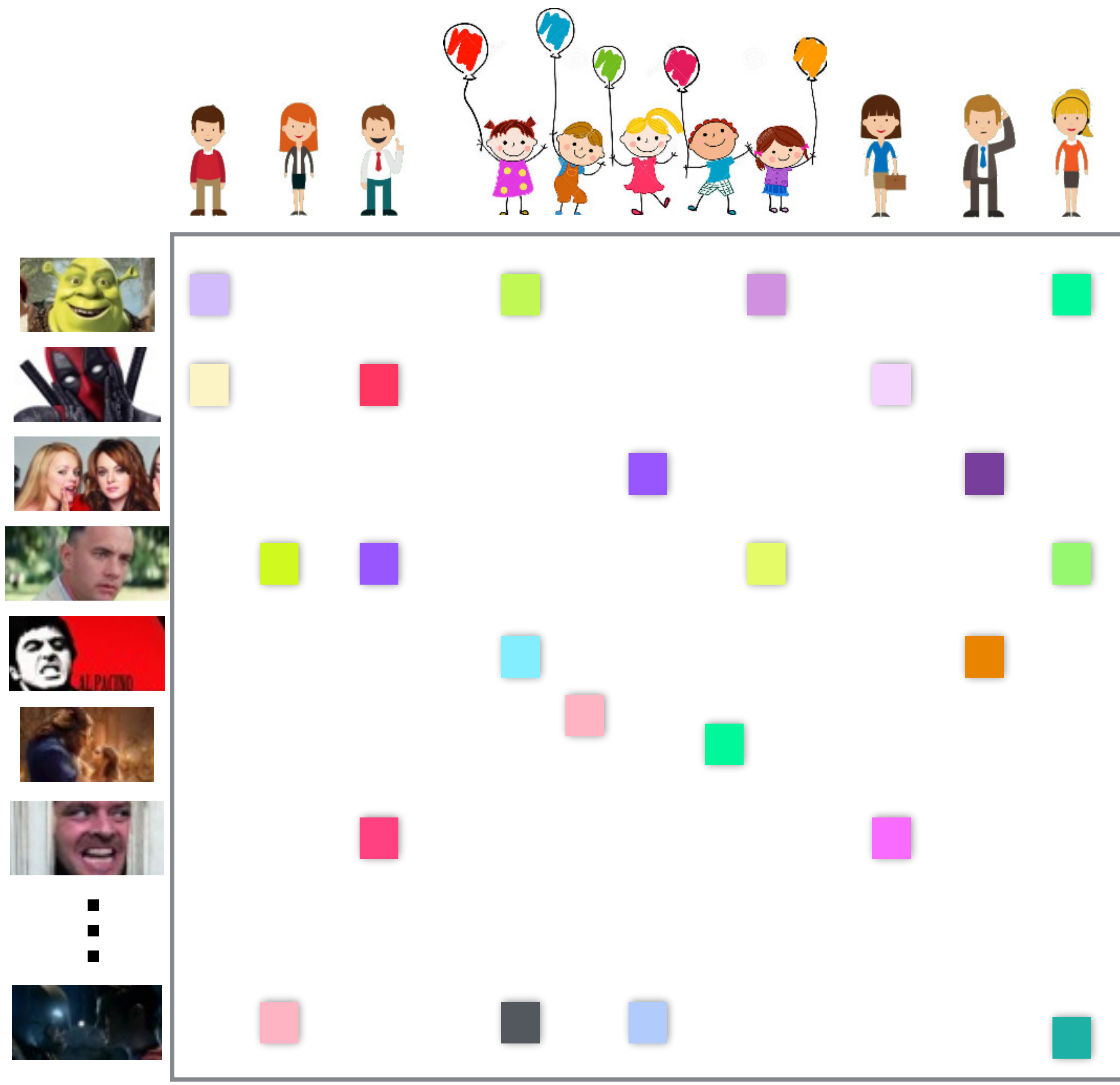
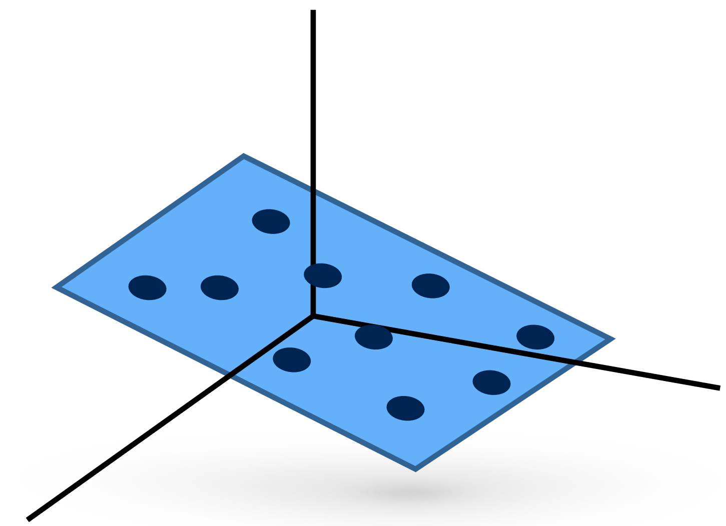
Column lies in
a subspace!



Columns lie in
a subspace!

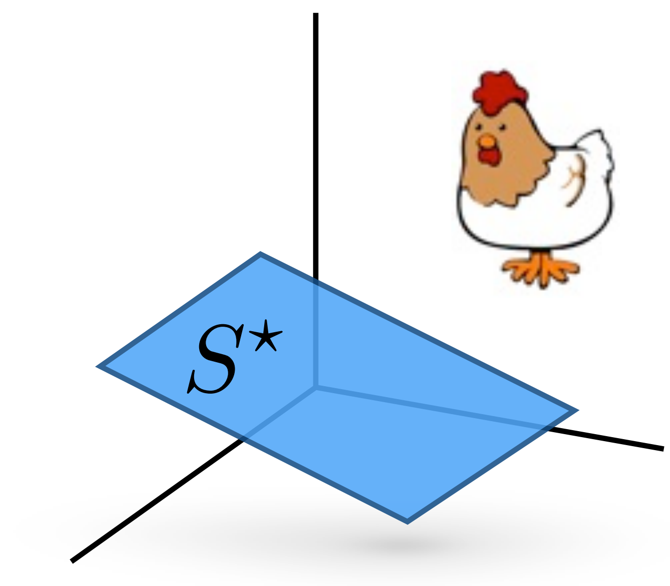


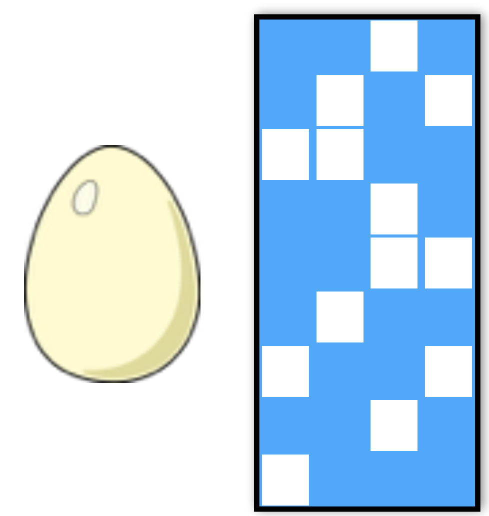
We want to find this Subspace!

Problem is: data is incomplete!

Chicken & Egg Problem

If I knew the subspace → I could find the missing values





I could find the subspace ← If I knew the missing values

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$$\min \ \|\mathbf{L}\|_*$$

s.t. $\|\mathbf{L}\|$ matches the observed entries

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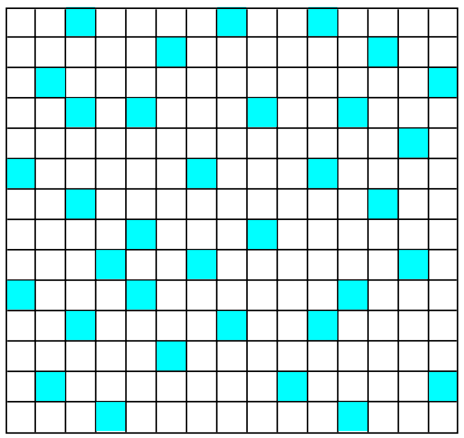

Uniform Sampling

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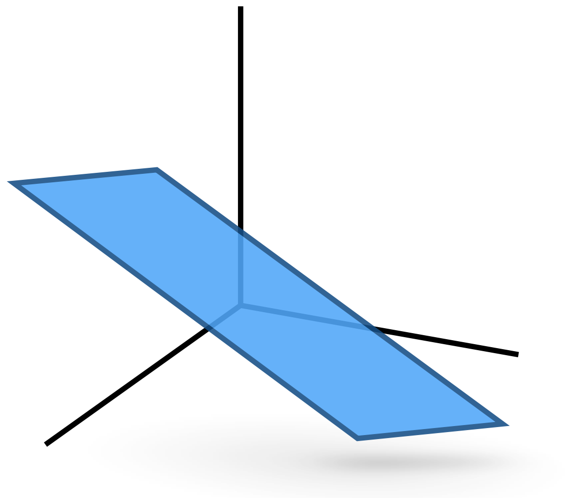
Incoherence



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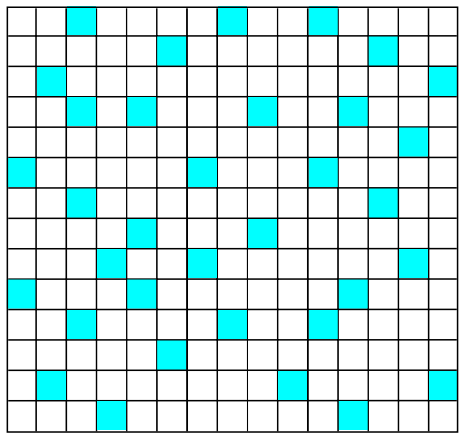

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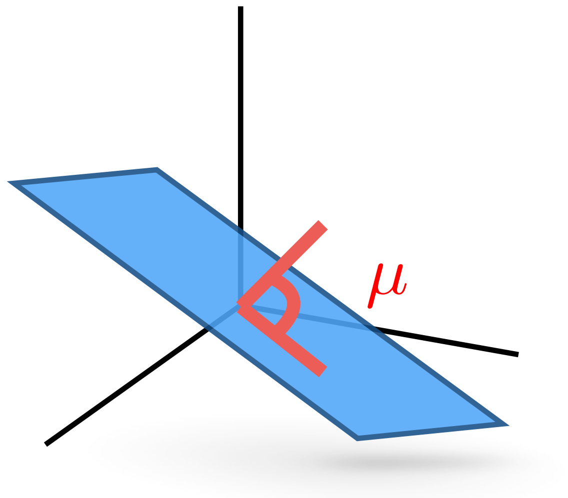

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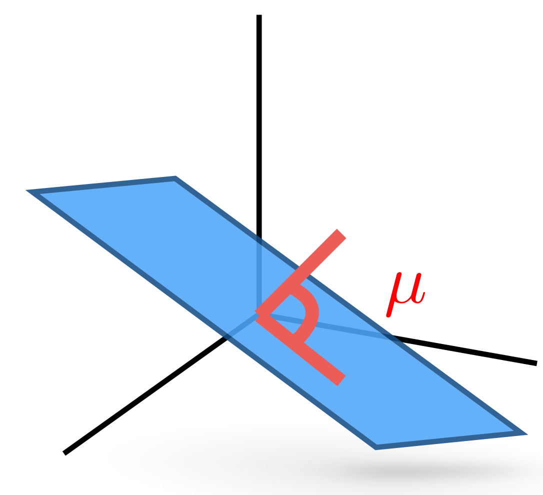

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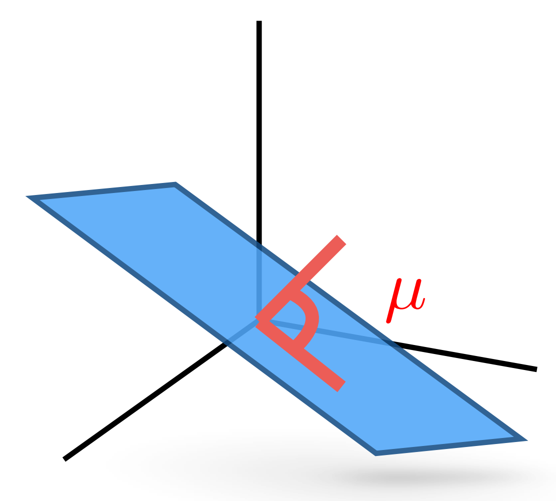

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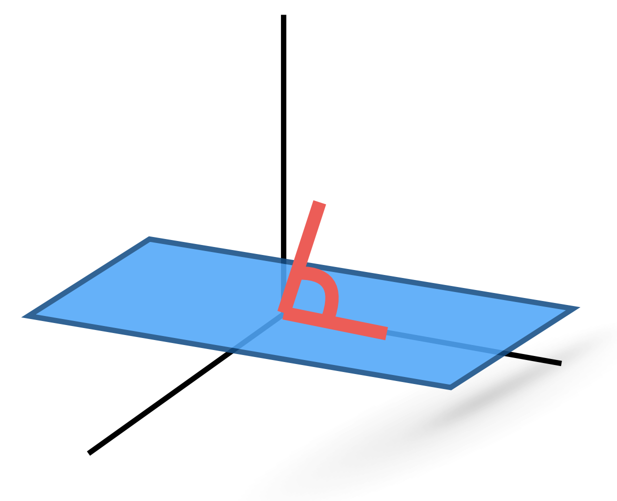

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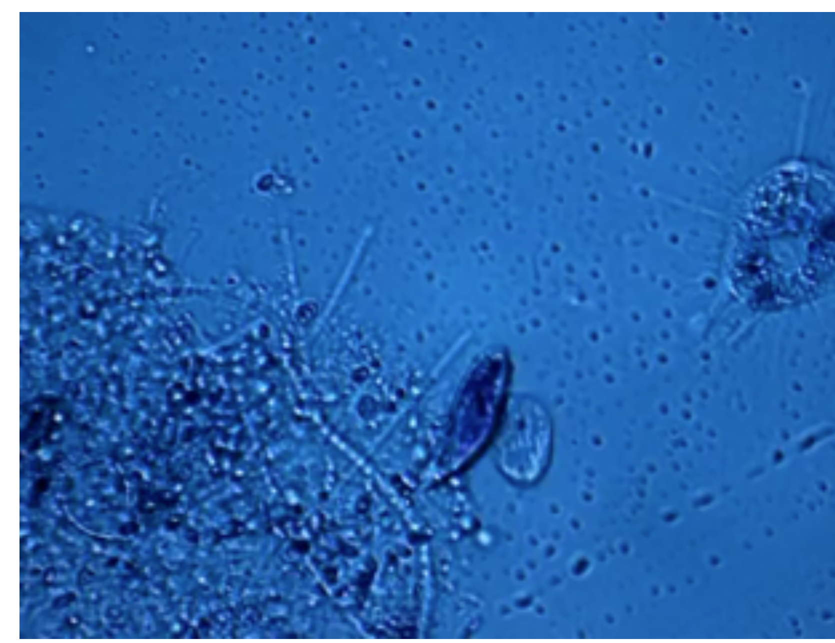

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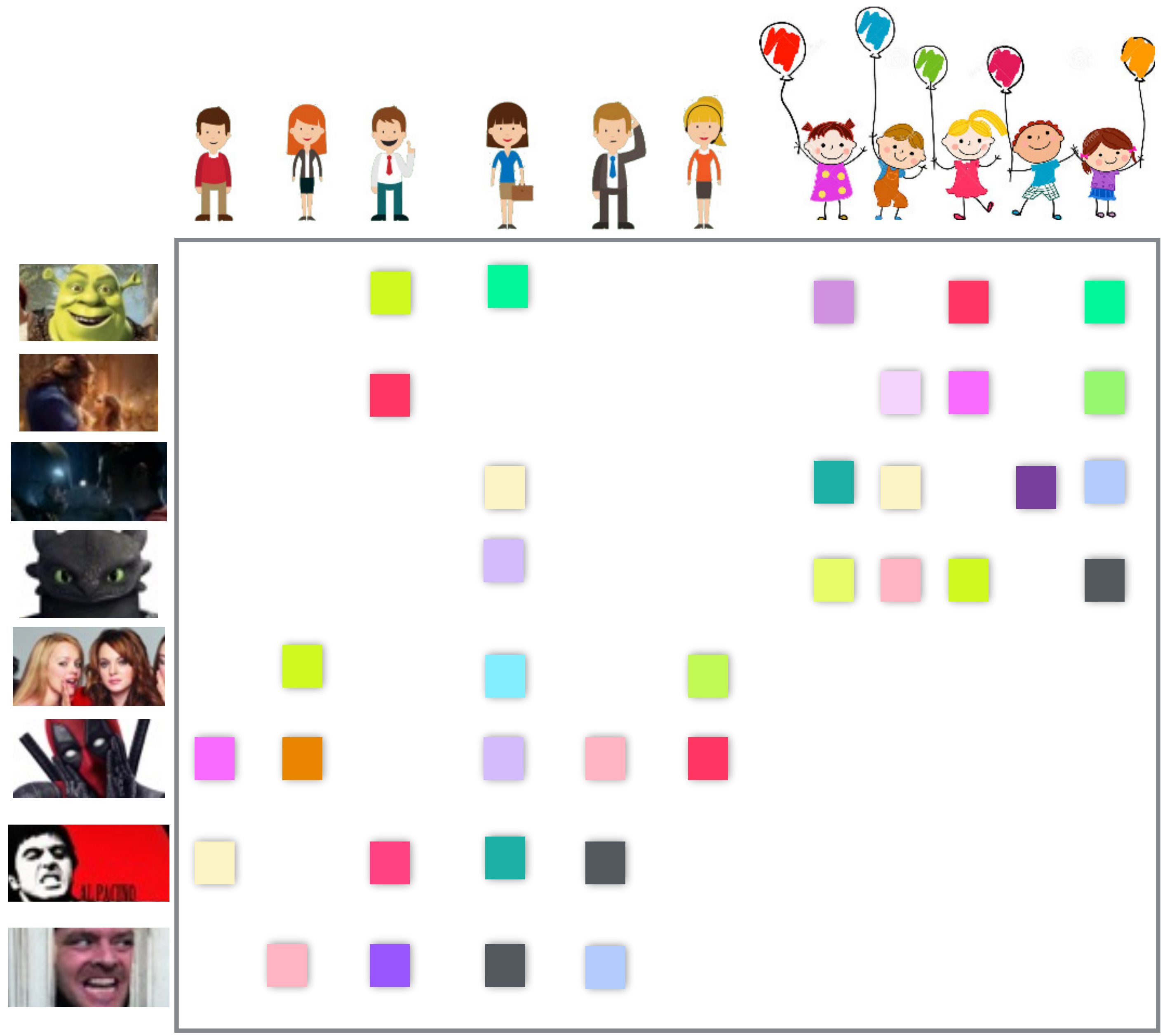
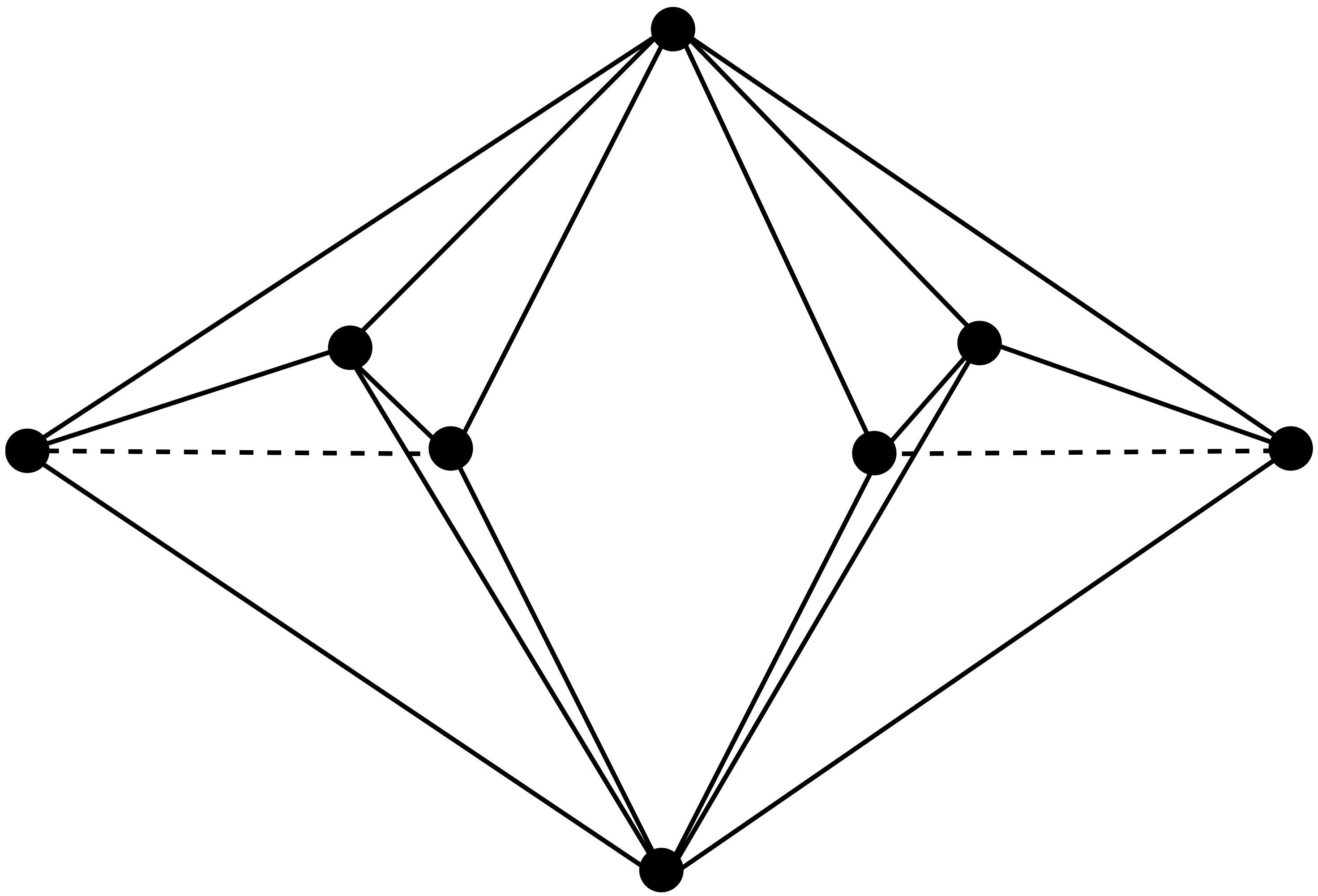


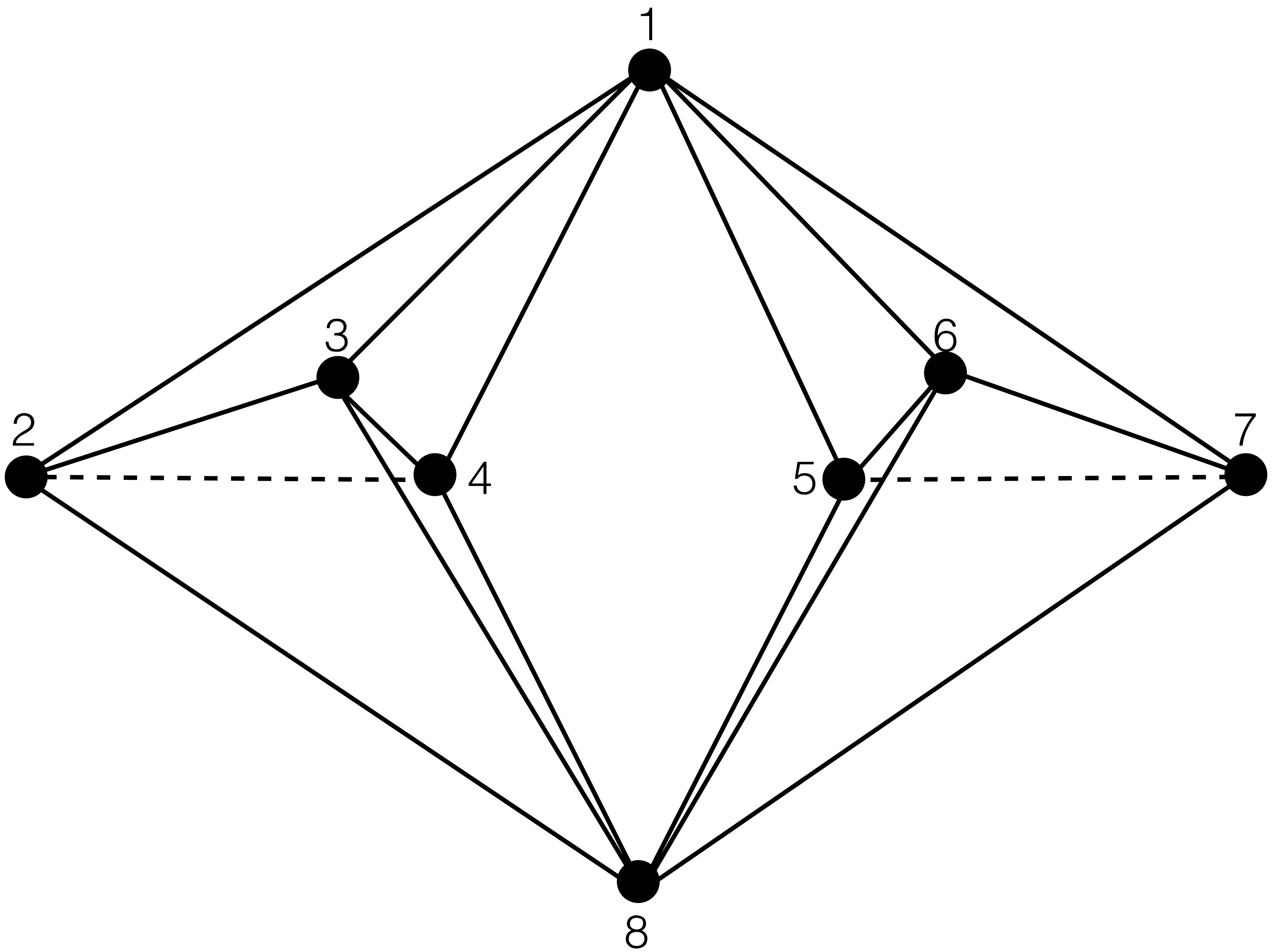
Recommender Systems

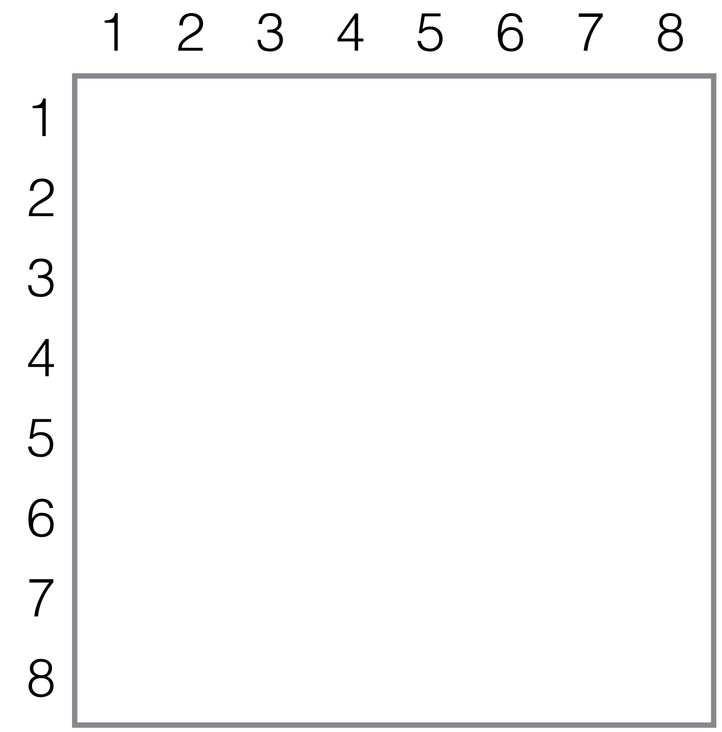
- Non-uniform Sampling!
- Coherent Subspace?



Rigidity and Graph Inference

- Non-uniform Sampling!
- Coherent Subspace!





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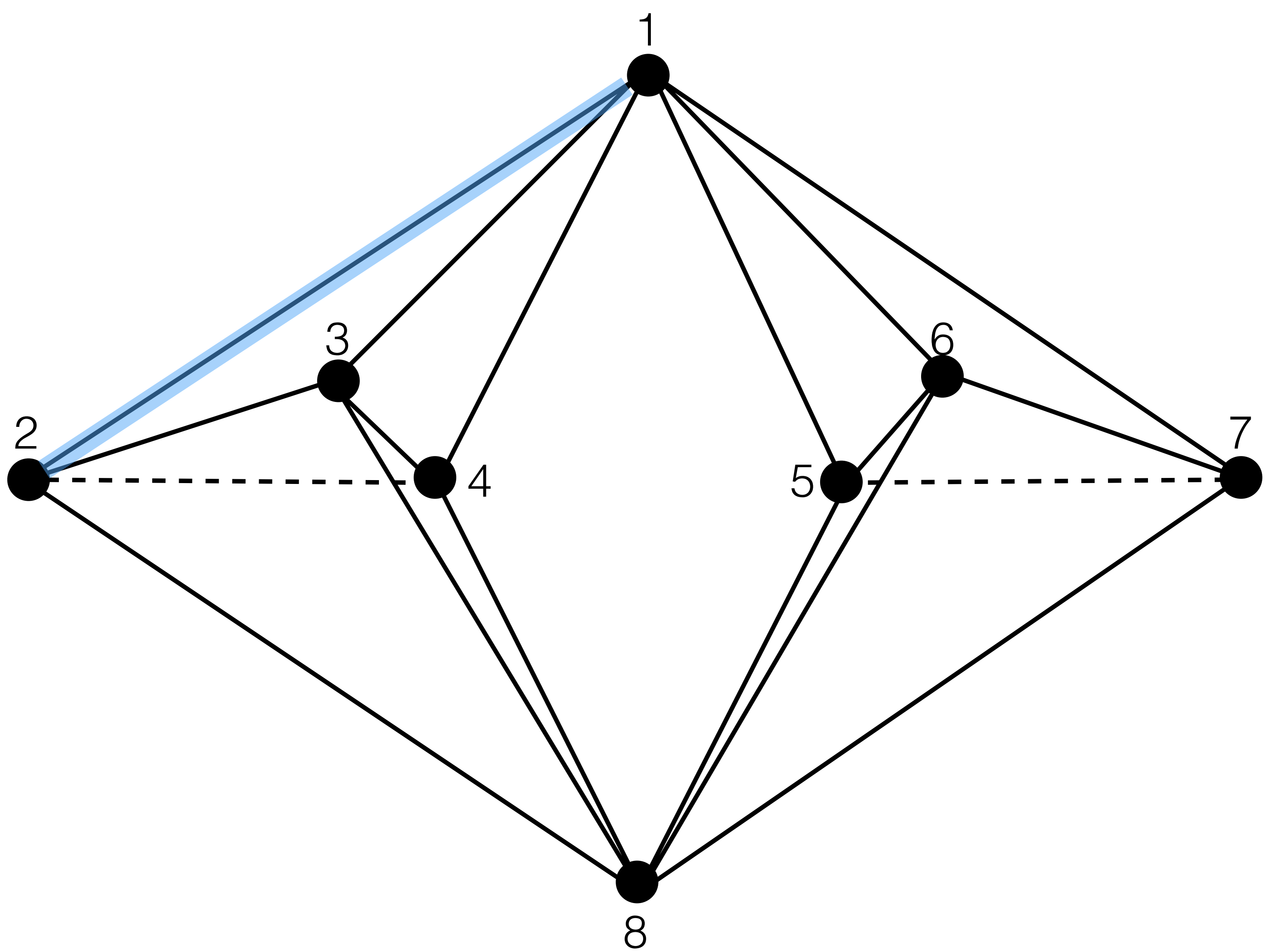
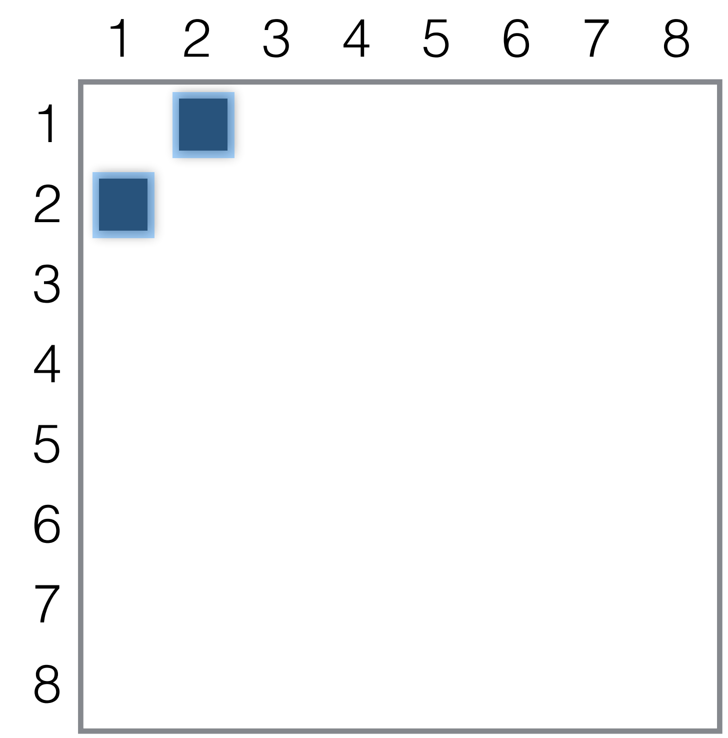
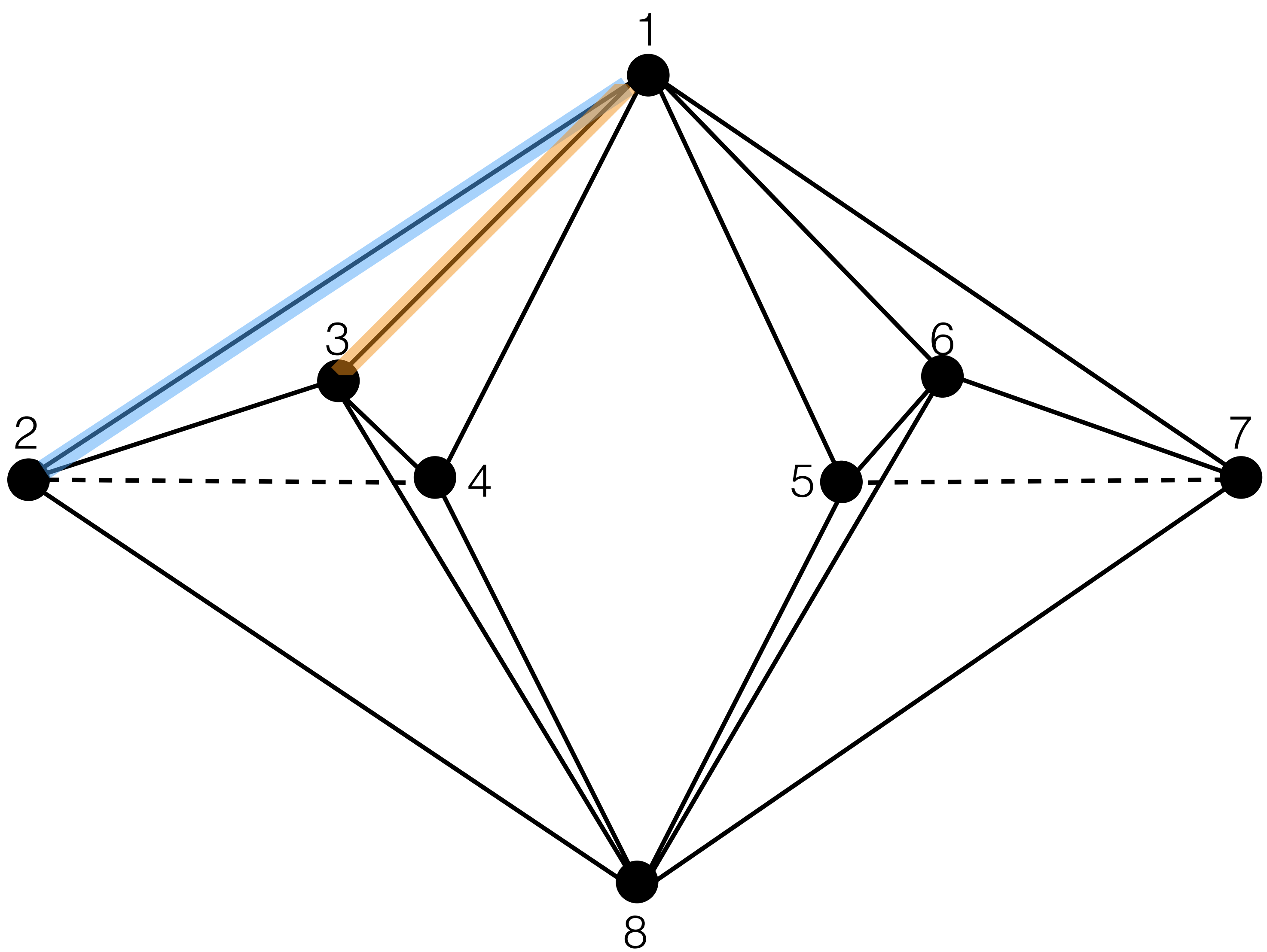
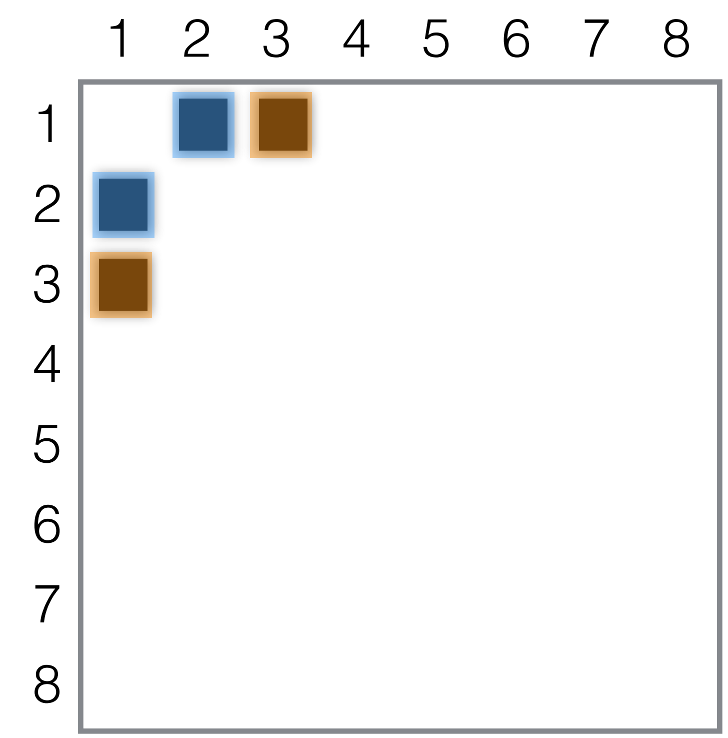




Rigidity and Graph Inference

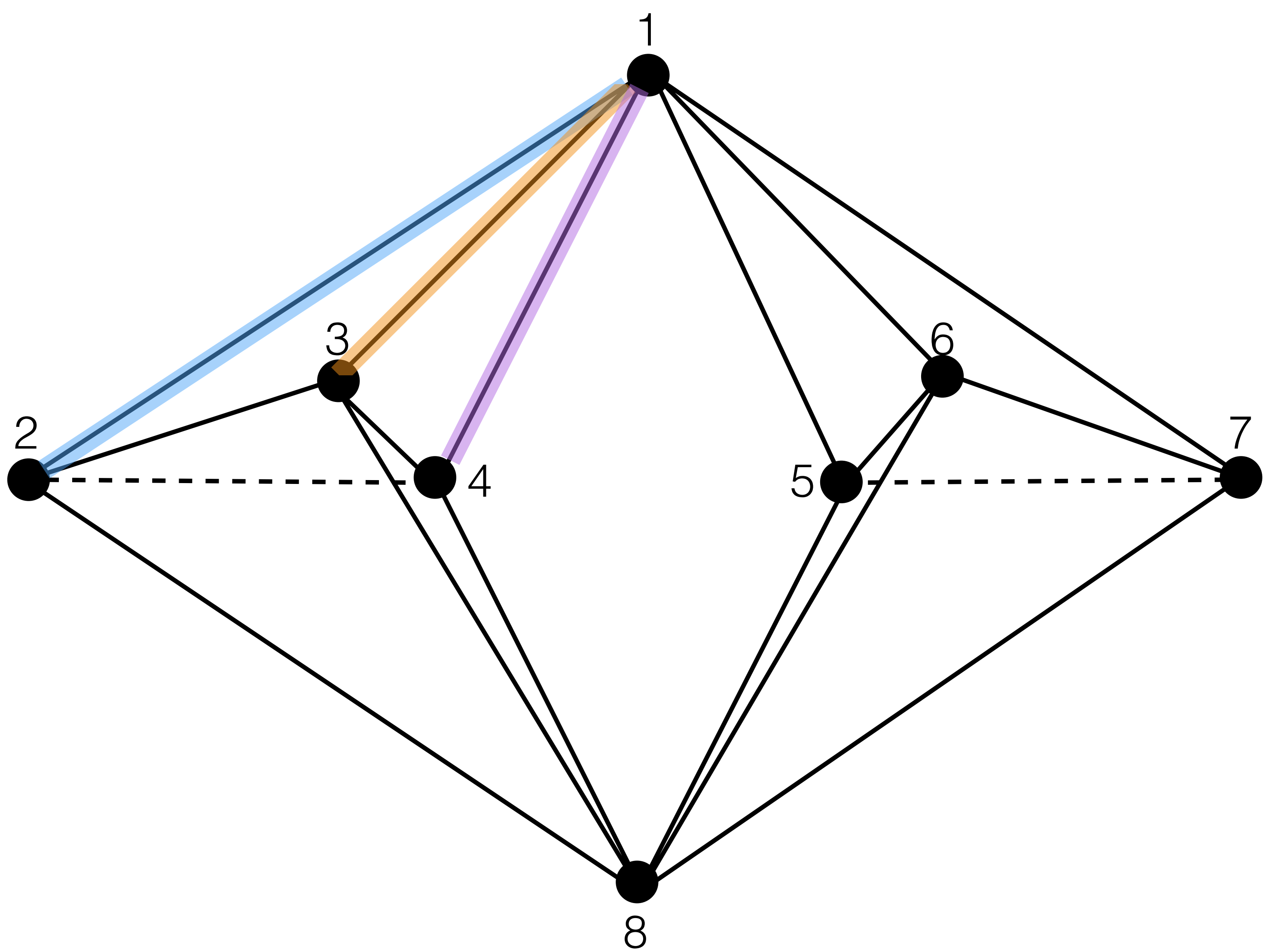
- Non-uniform Sampling!
- Coherent Subspace!

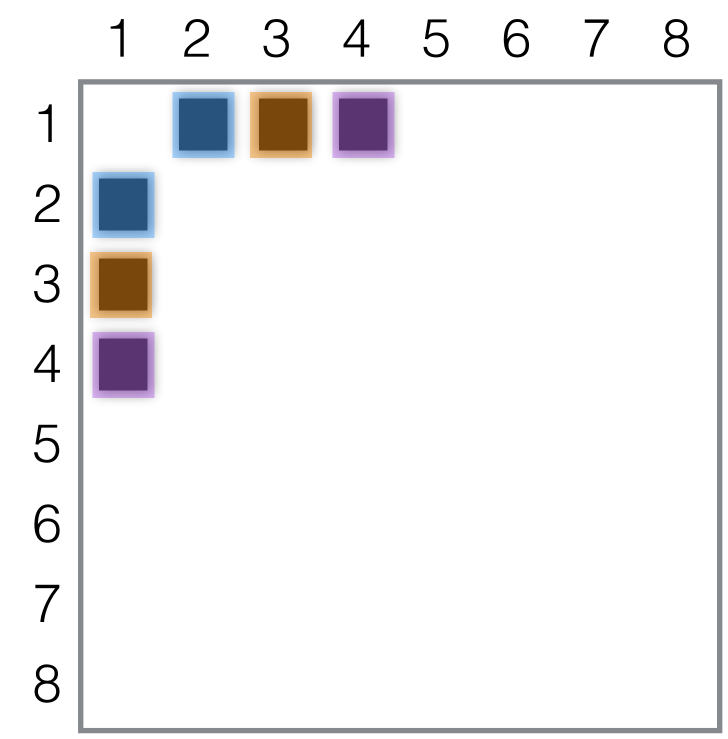




Rigidity and Graph Inference

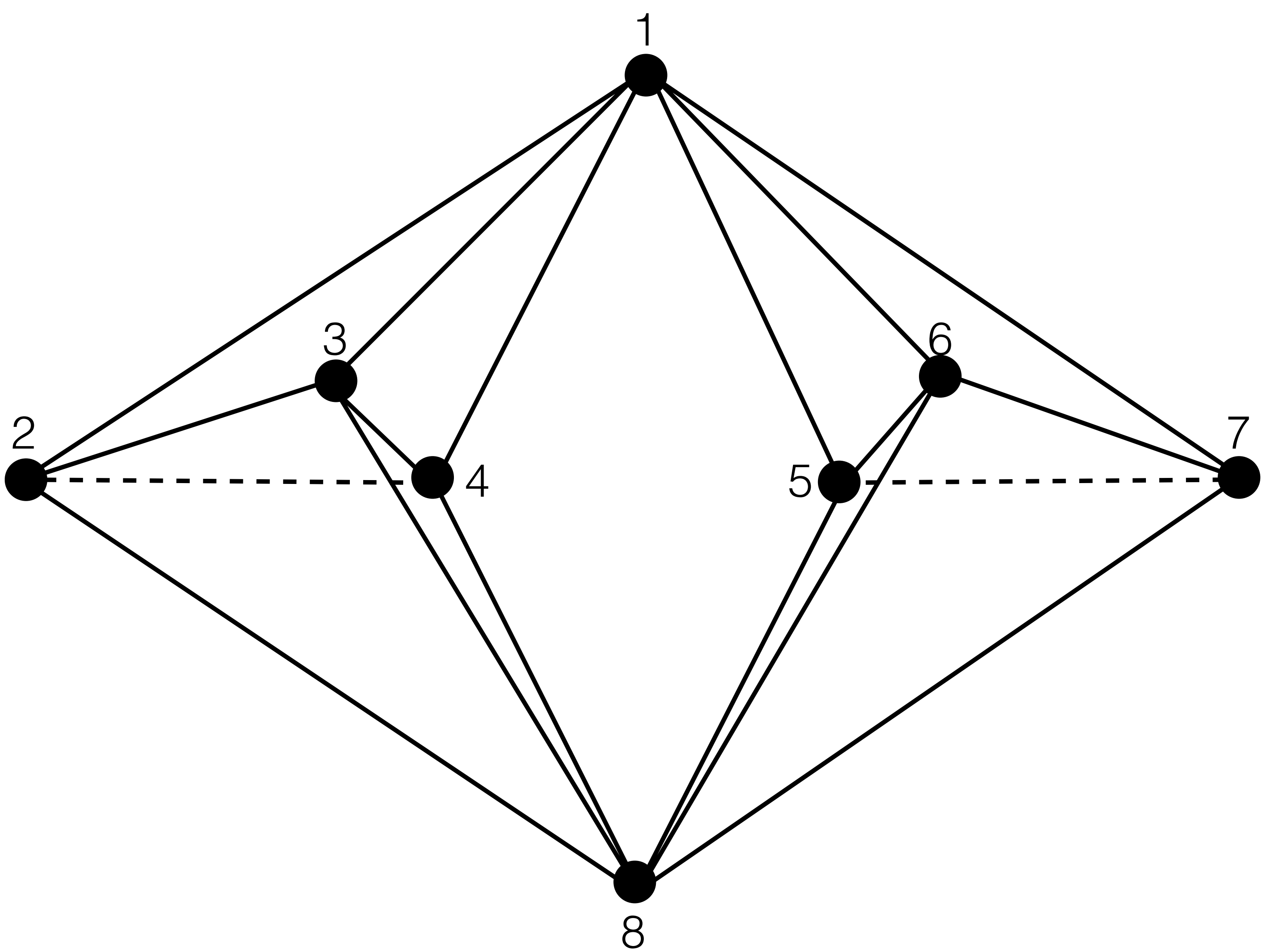
- Non-uniform Sampling!
- Coherent Subspace!

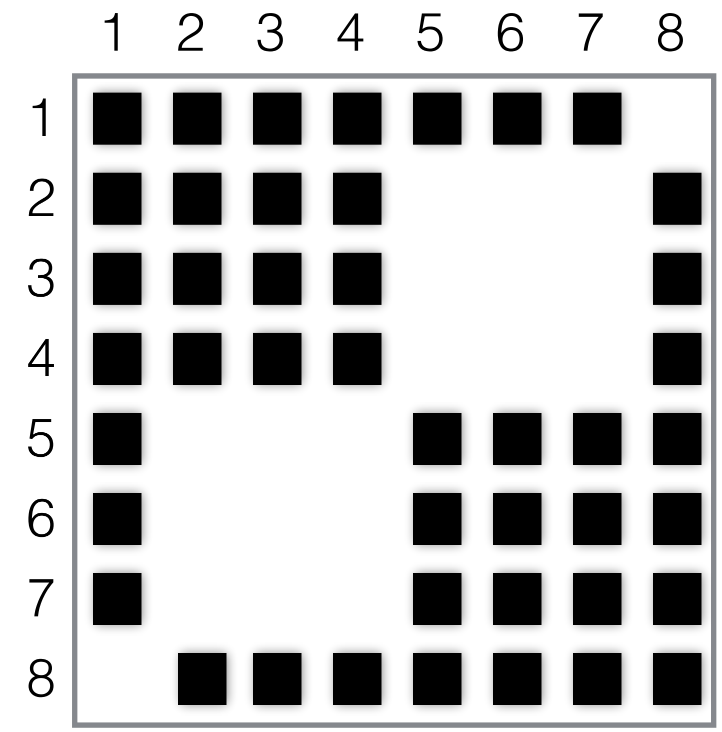




Rigidity and Graph Inference

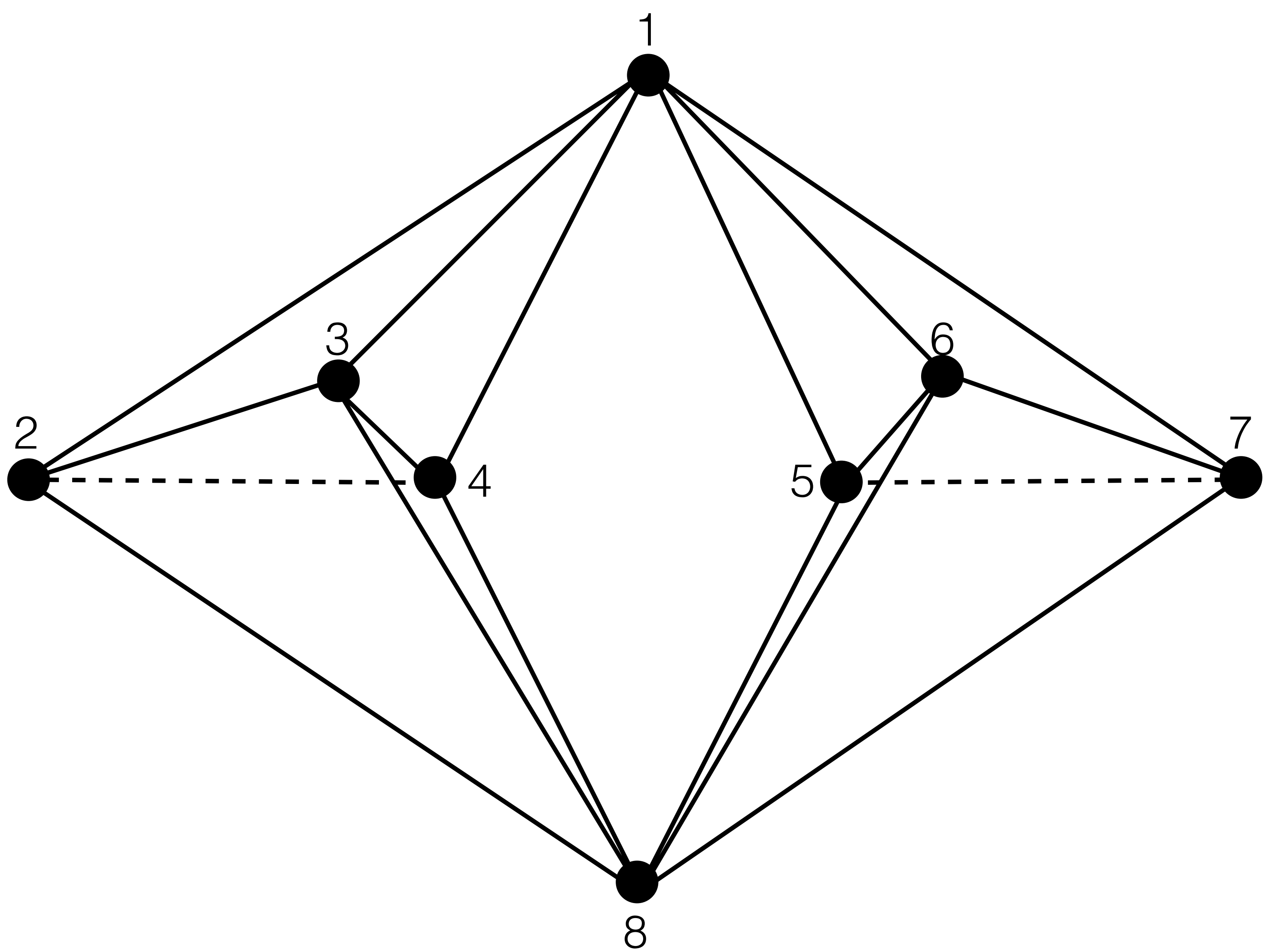
- Non-uniform Sampling!
- Coherent Subspace!

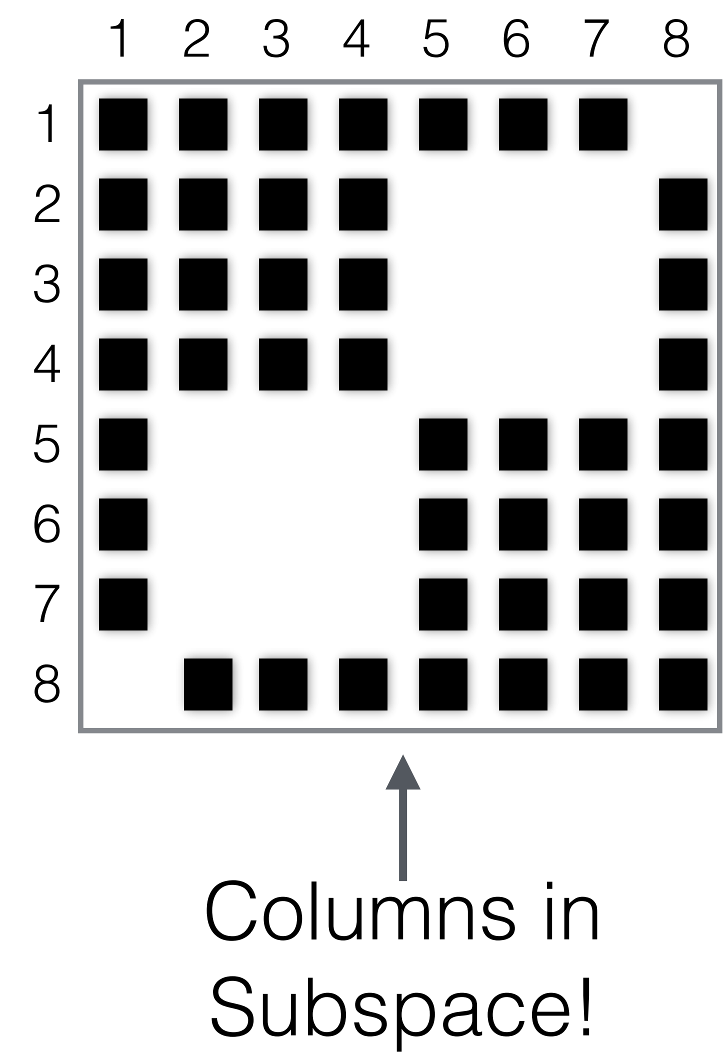




Rigidity and Graph Inference

- **Non-uniform Sampling!**
- **Coherent Subspace!**



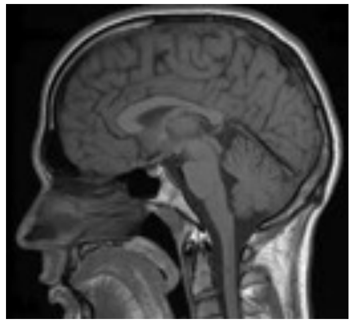


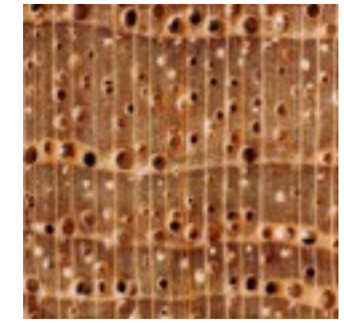
Rigidity and Graph Inference

- Non-uniform Sampling!
- Coherent Subspace!

Countless Applications

- **Non-uniform Sampling**
- **Coherent Subspace**



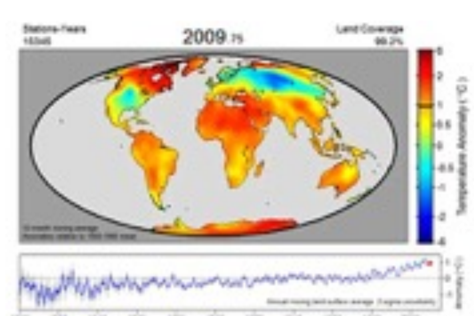



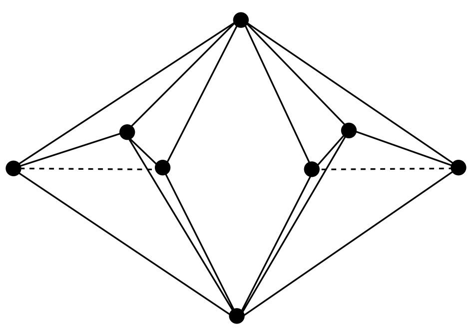
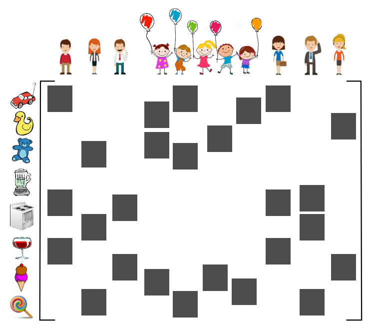


Countless Applications

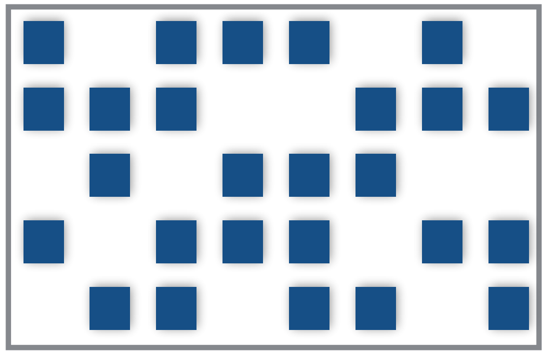
- **Non-uniform Sampling**
- **Coherent Subspace**



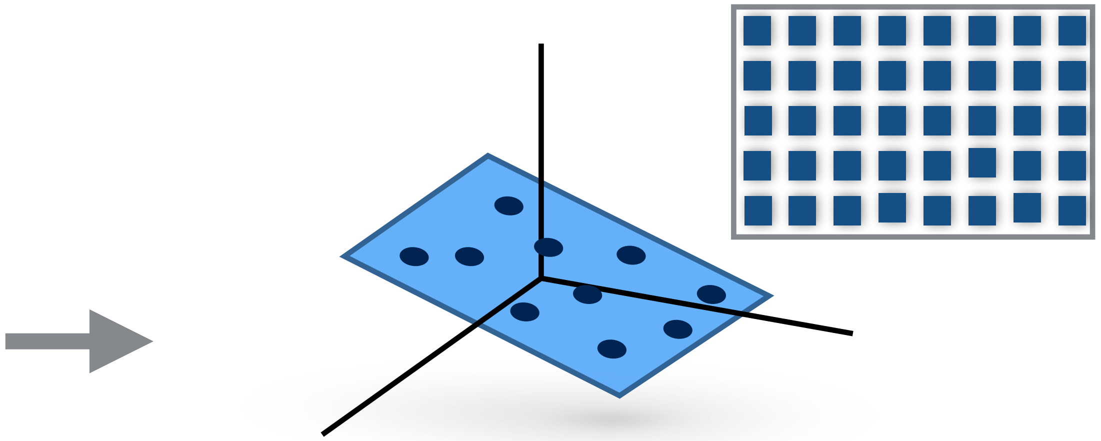


In general

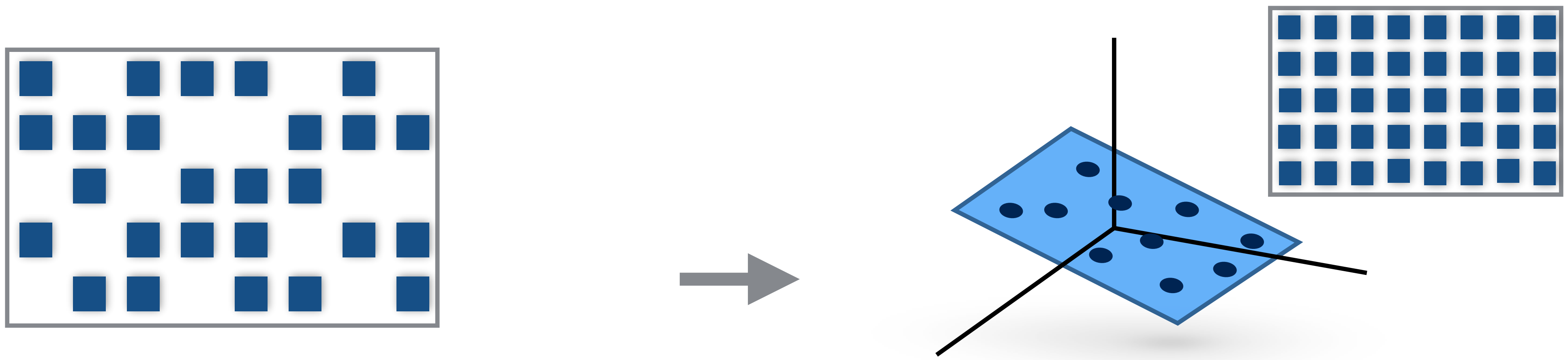


Given: incomplete data matrix



Can we find its subspace?

In general



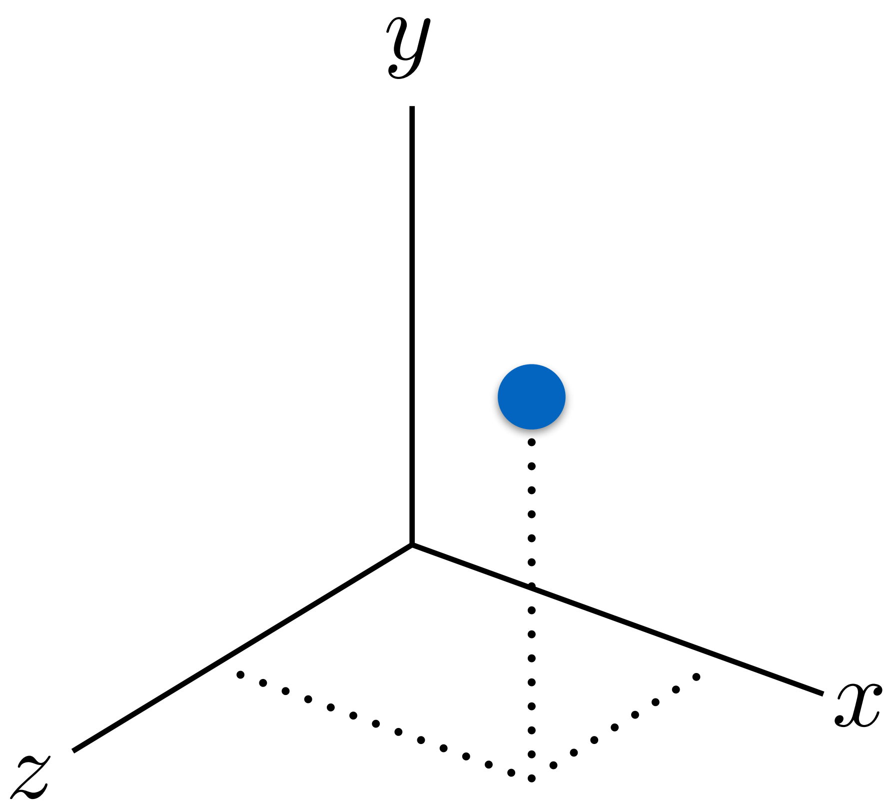
Given: incomplete data matrix **Can we find its subspace?**

To answer this:

Totally different way to think about the problem

- ~~Incoherence~~
- ~~Uniform~~
- ~~With high probability~~
- ~~Optimization~~

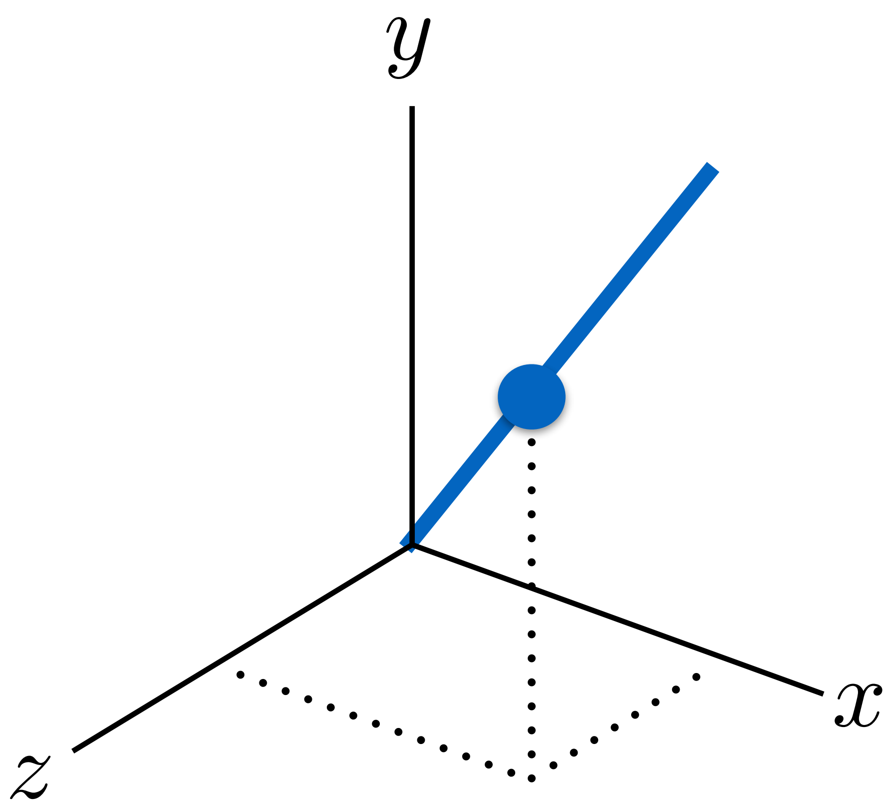
- Arbitrary
- Deterministic
- With probability 1
- Algebraic/Geometric



$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas

1-dimensional subspace, 1 data point



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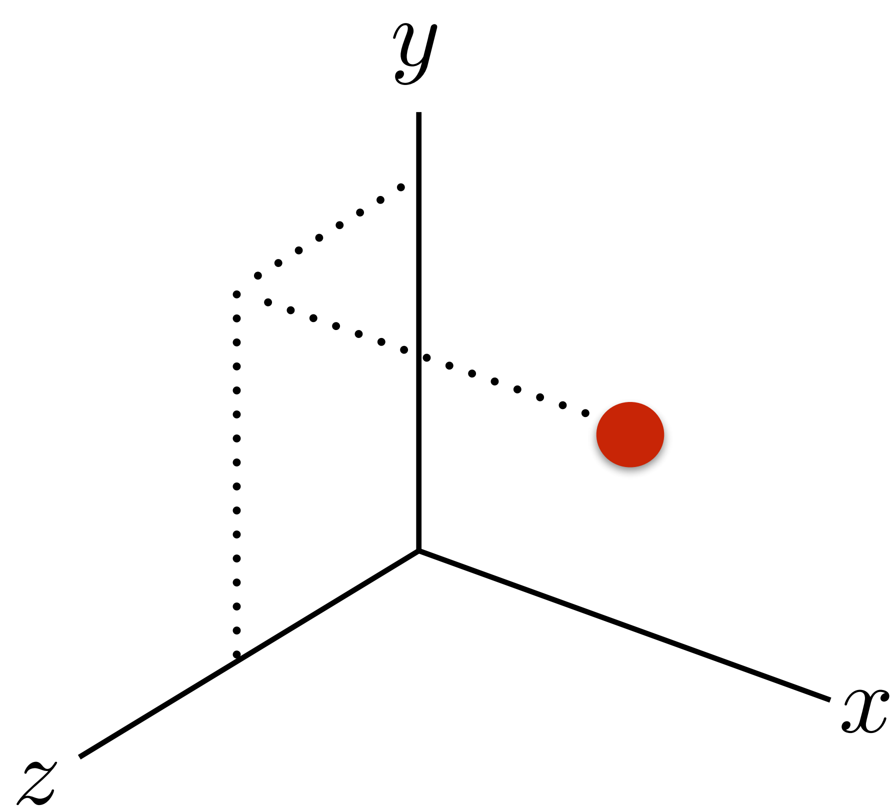
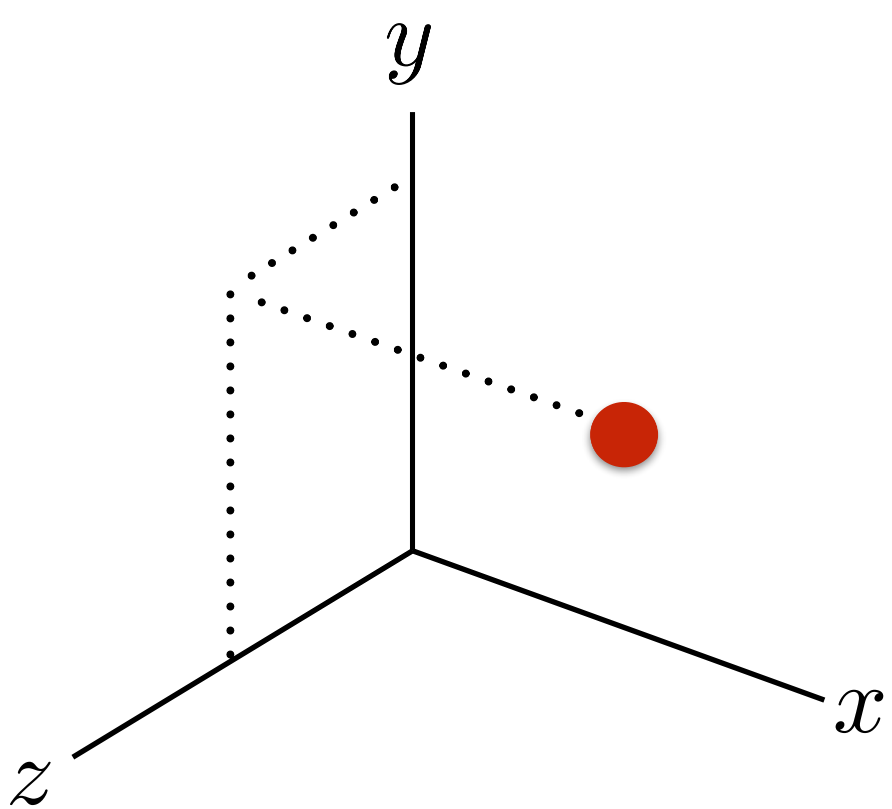
1-dimensional subspace, 1 data point



$$\begin{bmatrix} \cdot \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas

1-dimensional subspace, 1 **incomplete** data point



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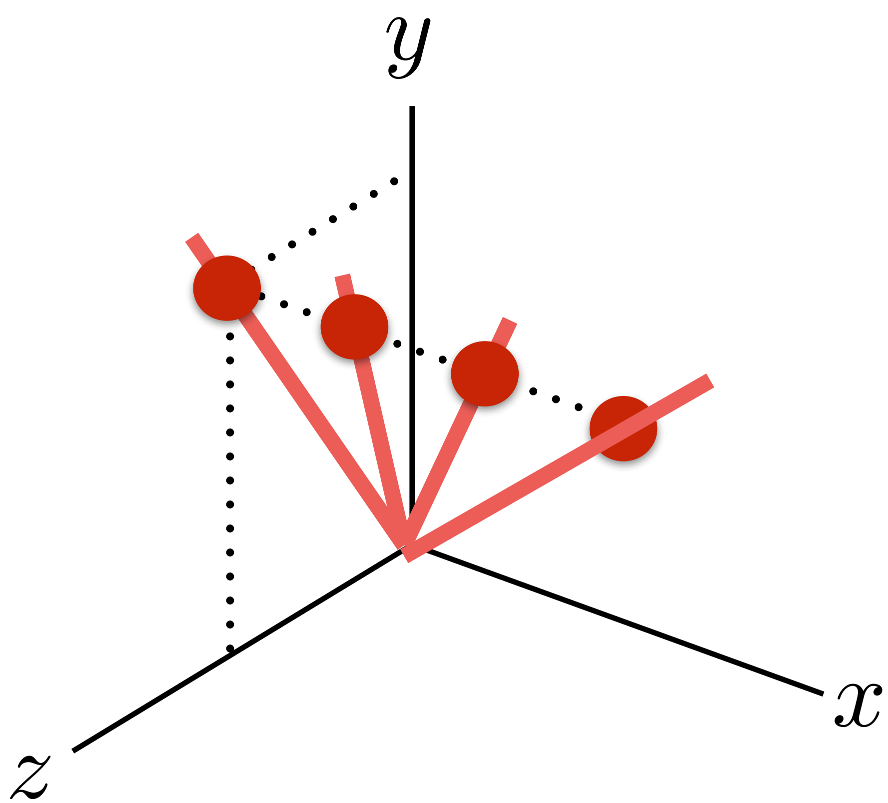

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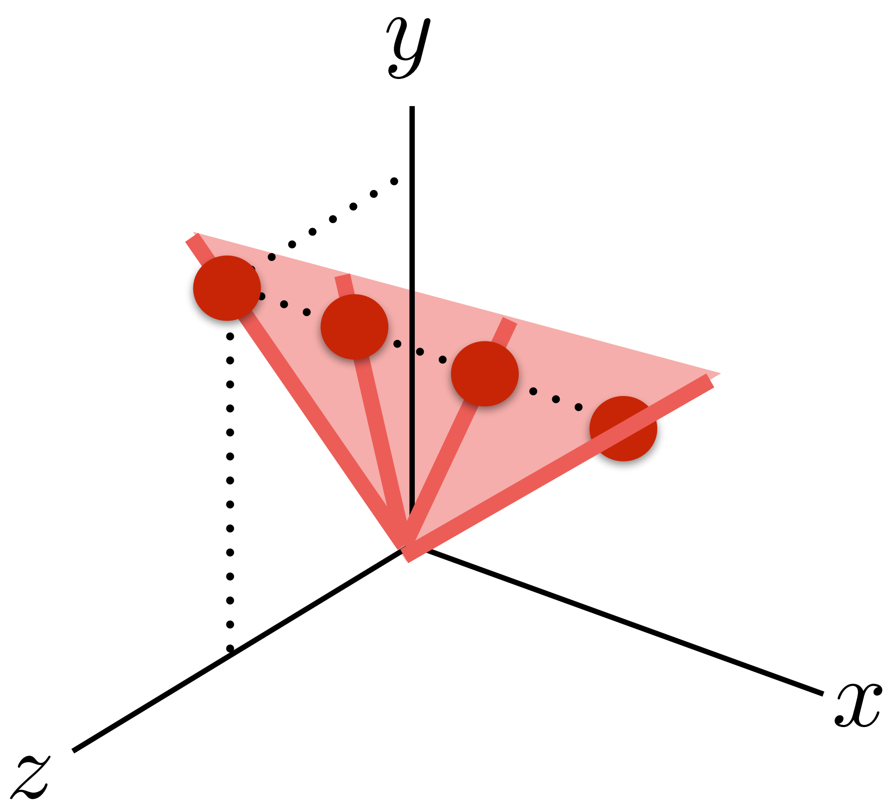
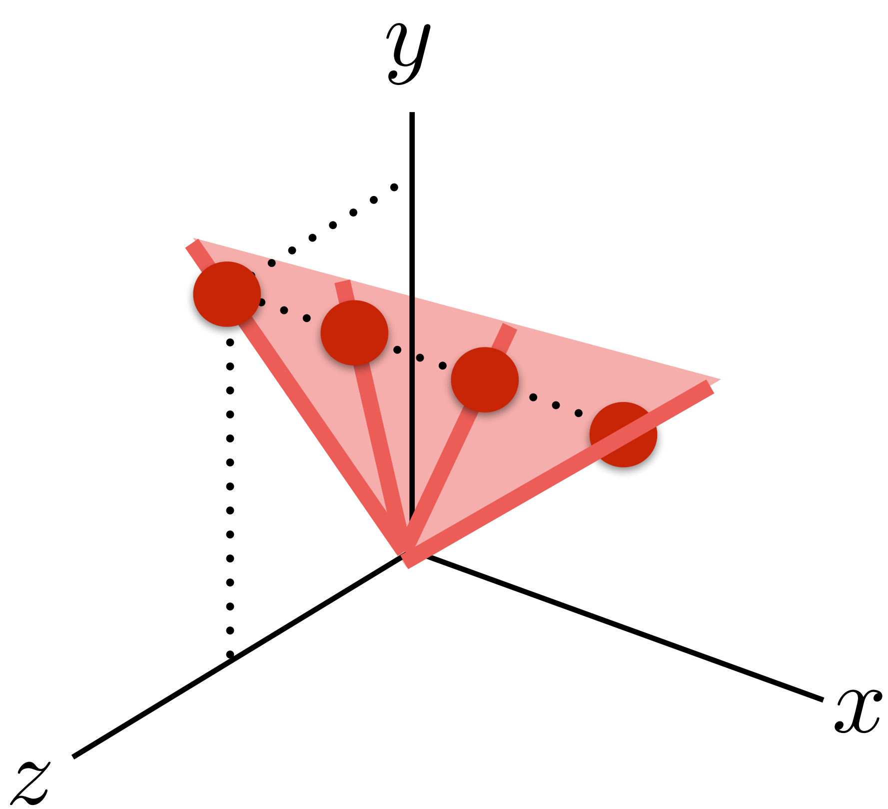


$$\begin{bmatrix} \cdot \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas

1-dimensional subspace, 1 **incomplete** data point

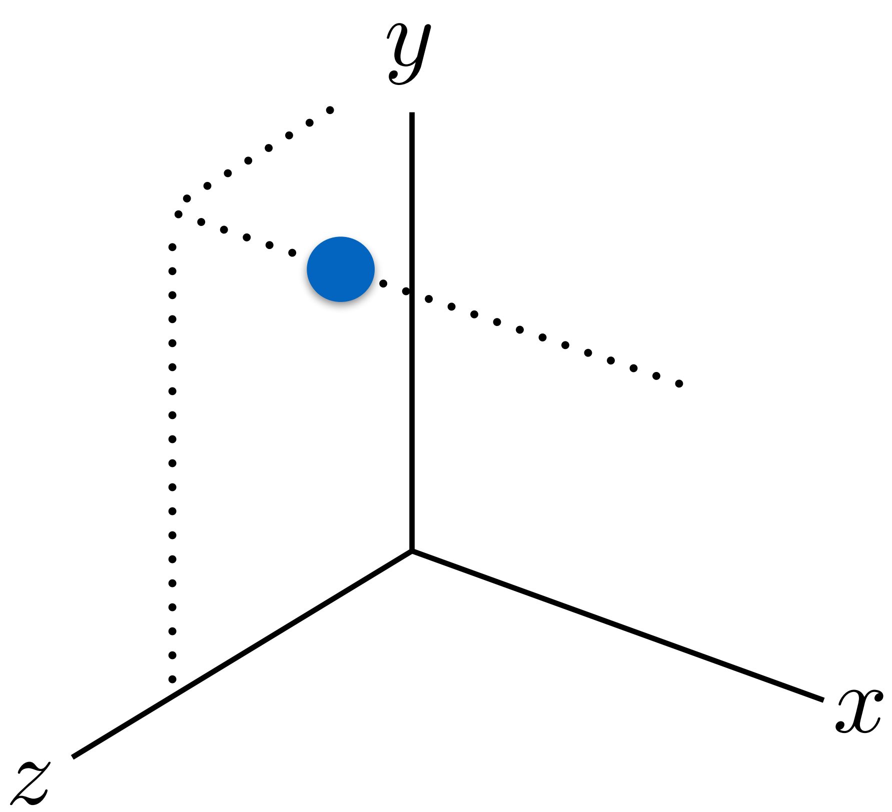


This column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas

1-dimensional subspace, 1 **incomplete** data point

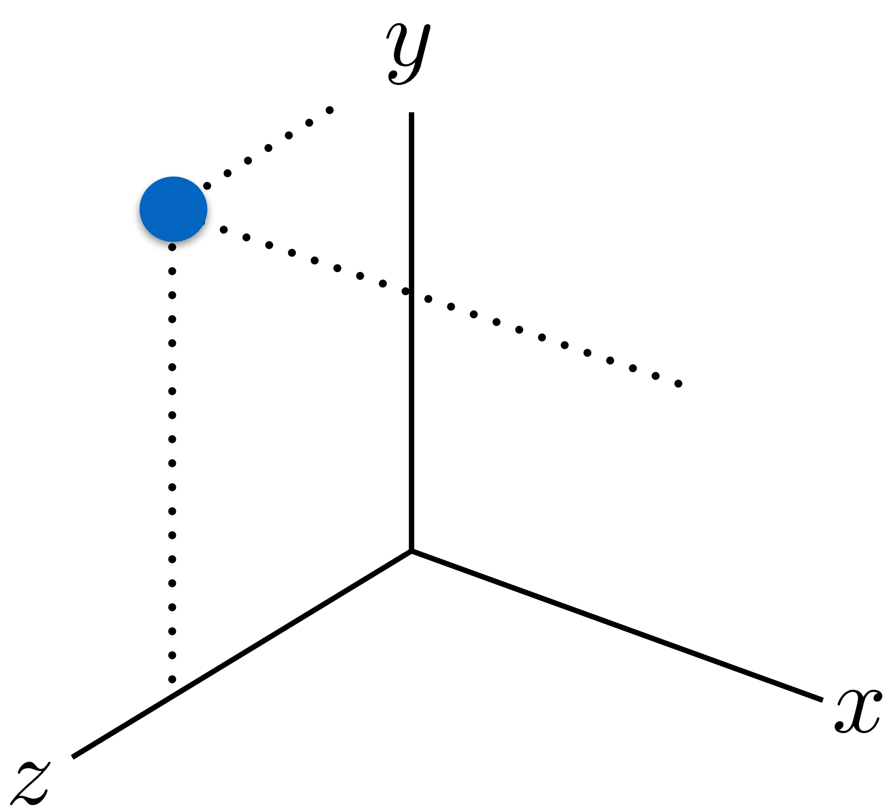


New column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & \cdot \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$$

A flavor of our ideas

1-dimensional subspace, 2 **incomplete** data points



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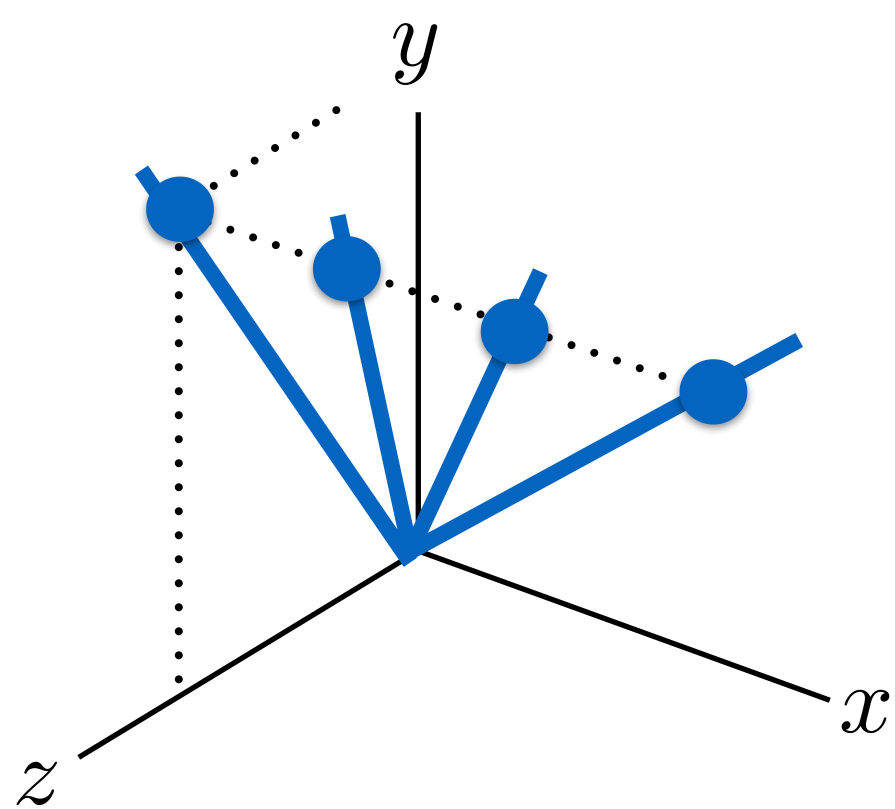
A flavor of our ideas

1-dimensional subspace, 2 **incomplete** data points



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A flavor of our ideas

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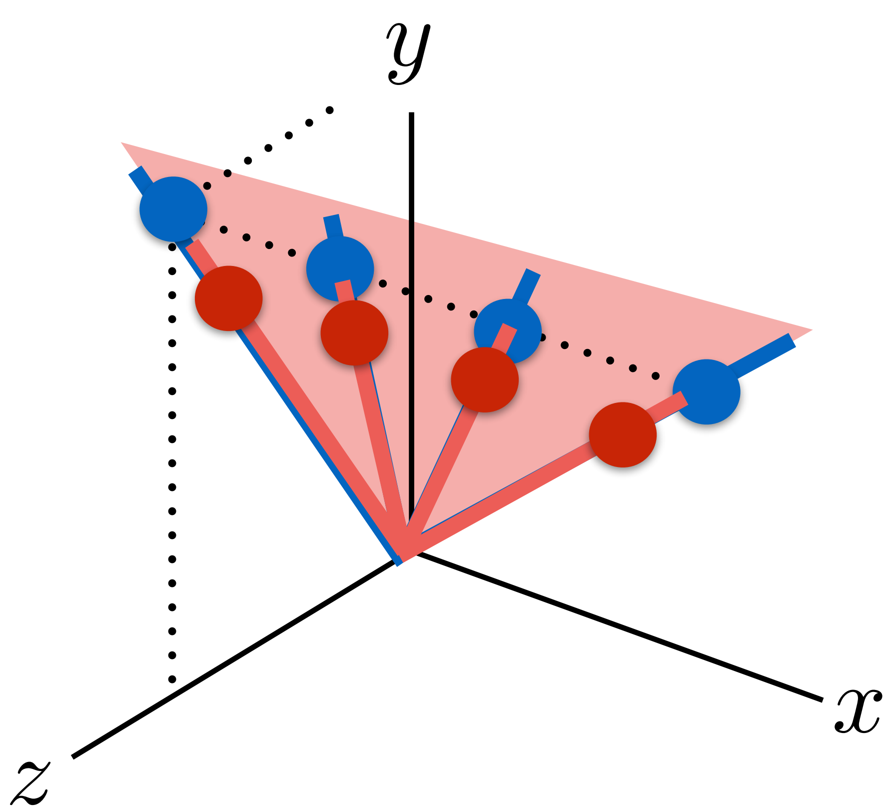

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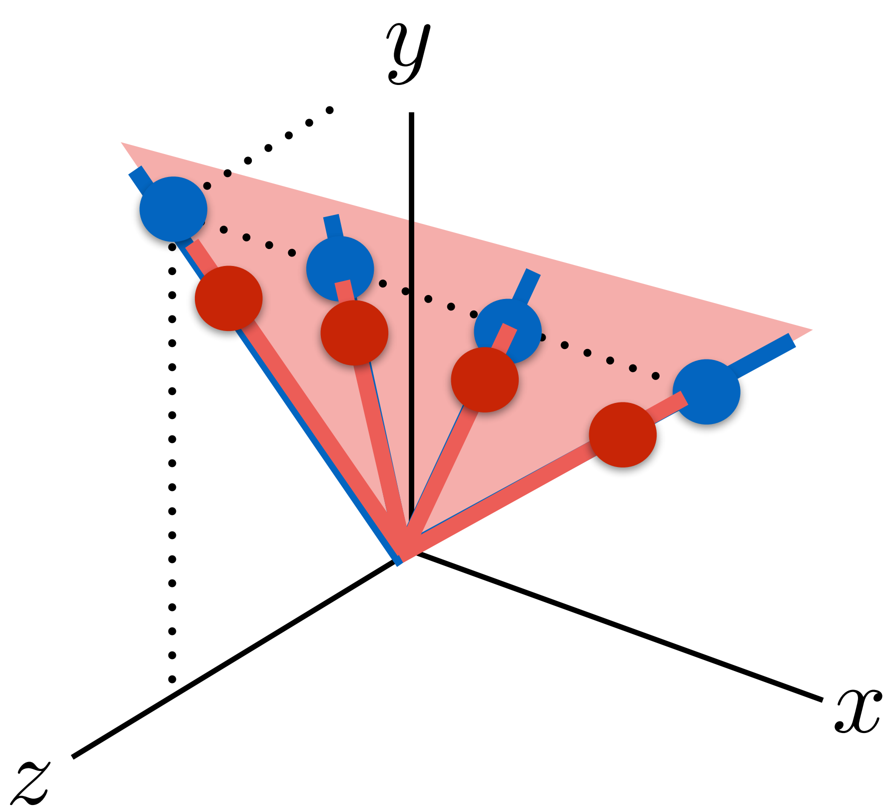


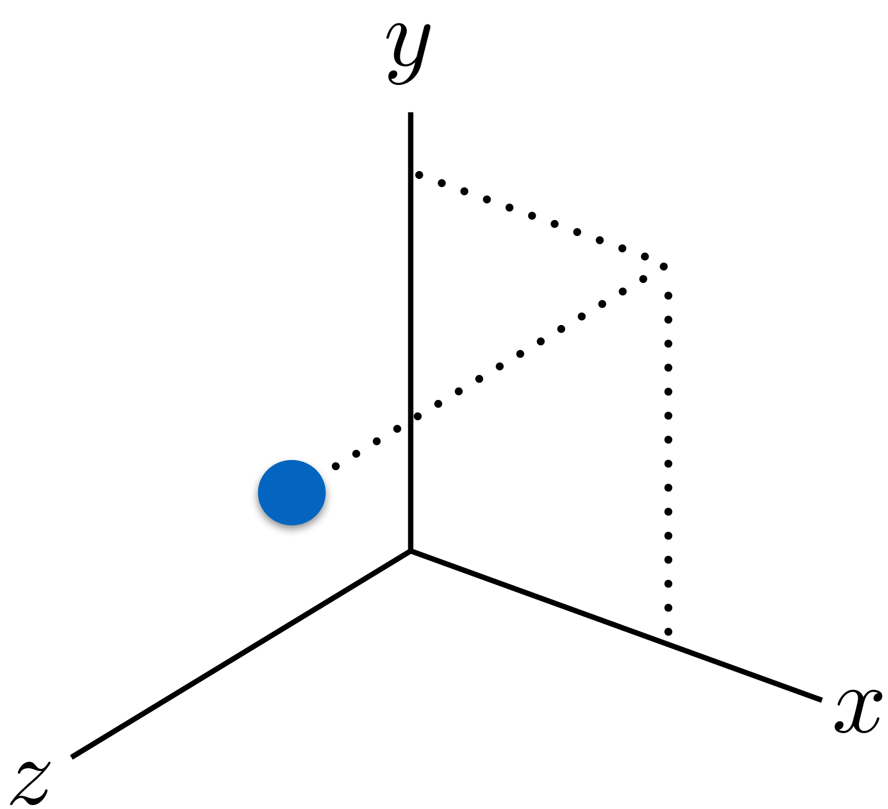
New column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & \cdot \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$$

New restriction may be redundant!

A flavor of our ideas

1-dimensional subspace, 2 **incomplete** data points



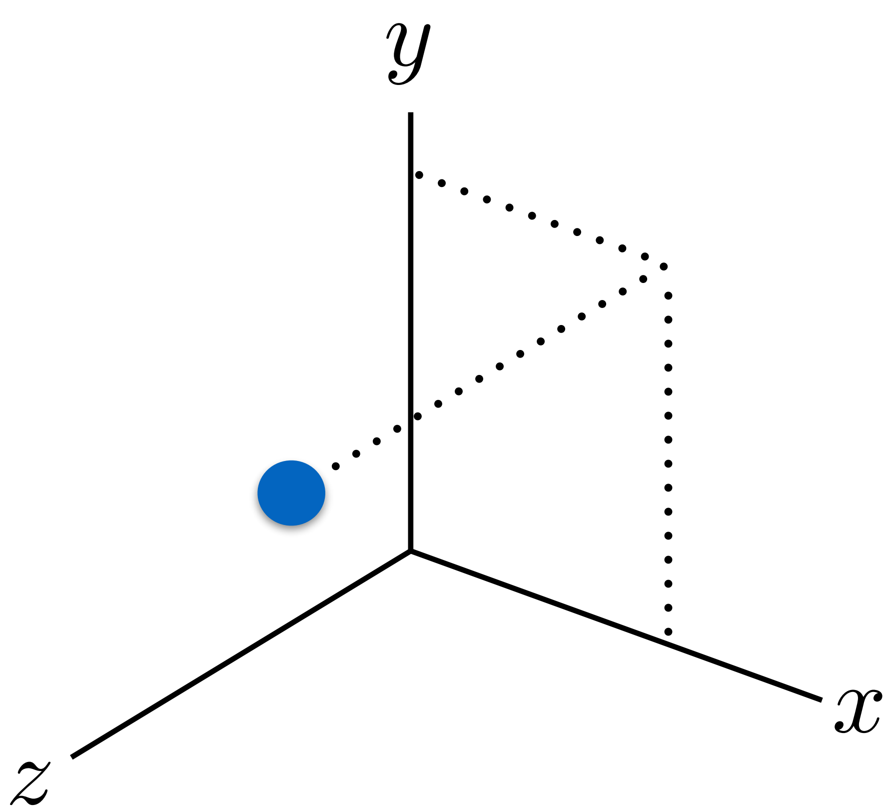
New column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

New restriction may be redundant!

A flavor of our ideas

1-dimensional subspace, 2 **incomplete** data points



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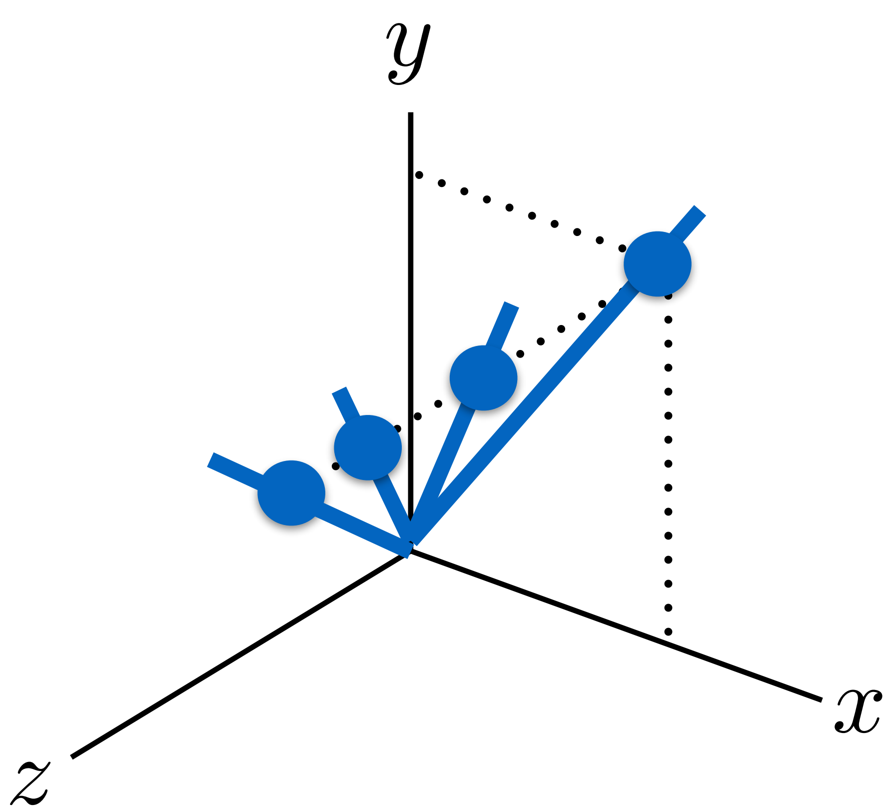

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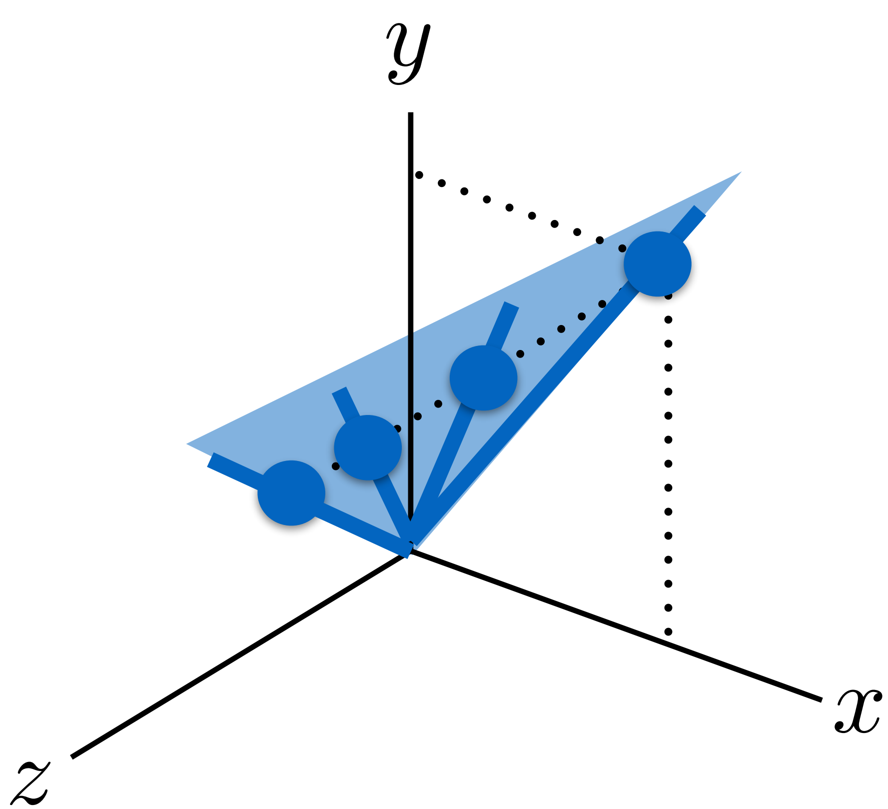


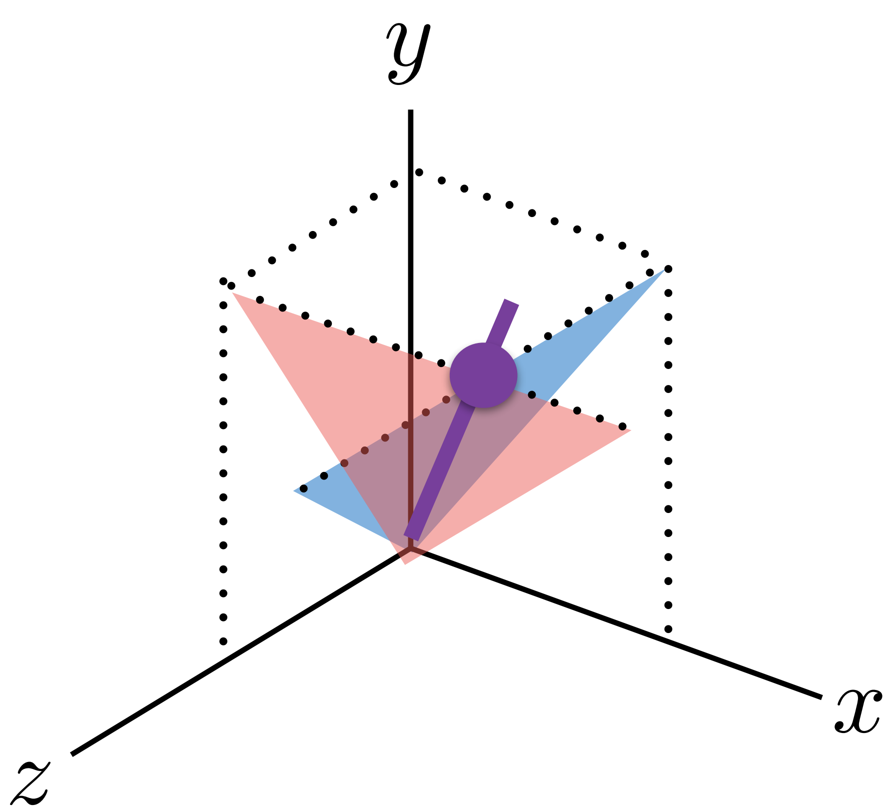
New column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

New restriction may be redundant!

A flavor of our ideas

1-dimensional subspace, 2 **incomplete** data points



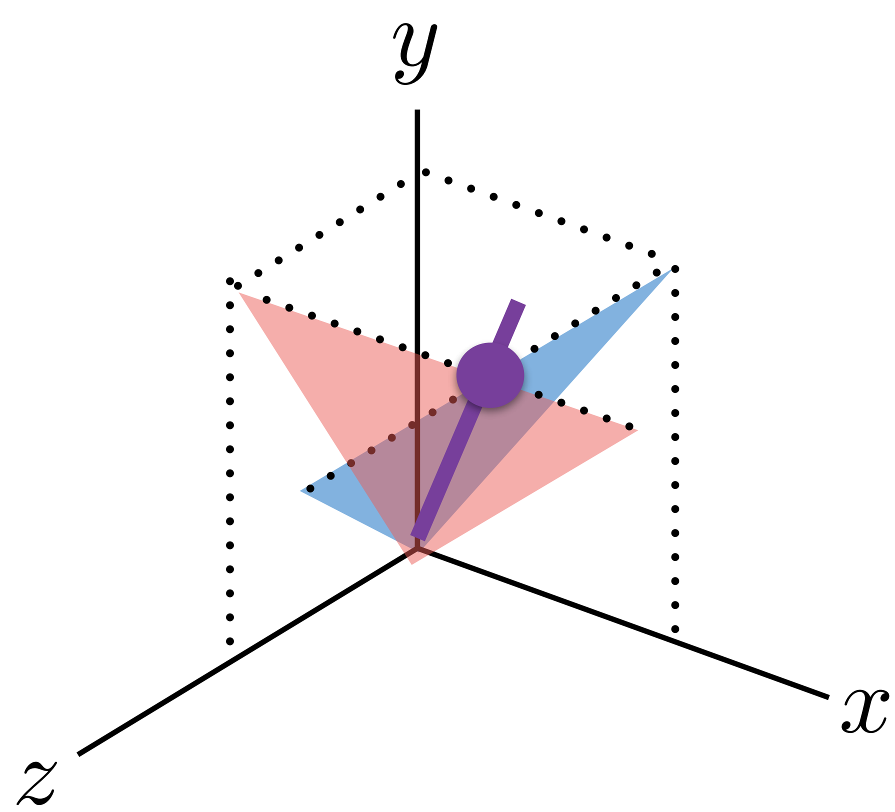
Each column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

New restriction may be redundant!

A flavor of our ideas

1-dimensional subspace, 2 **incomplete** data points



Each column imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

New restriction may be redundant!

Depends on which entries we observe!

A flavor of our ideas

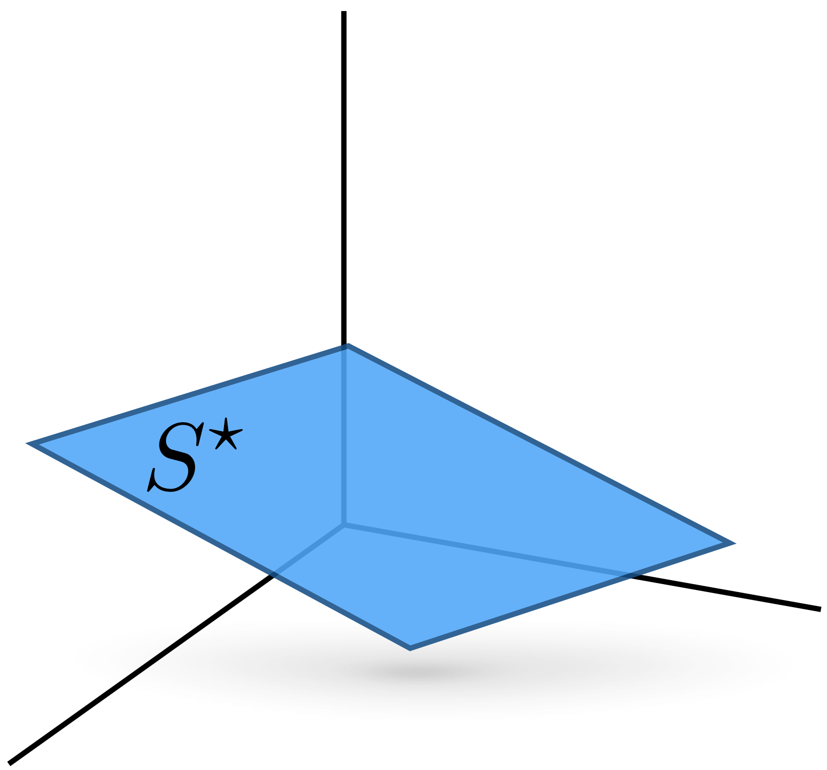
1-dimensional subspace, 2 **incomplete** data points

Determines Exactly:
Which entries you need to observe
to find a subspace

Our main result

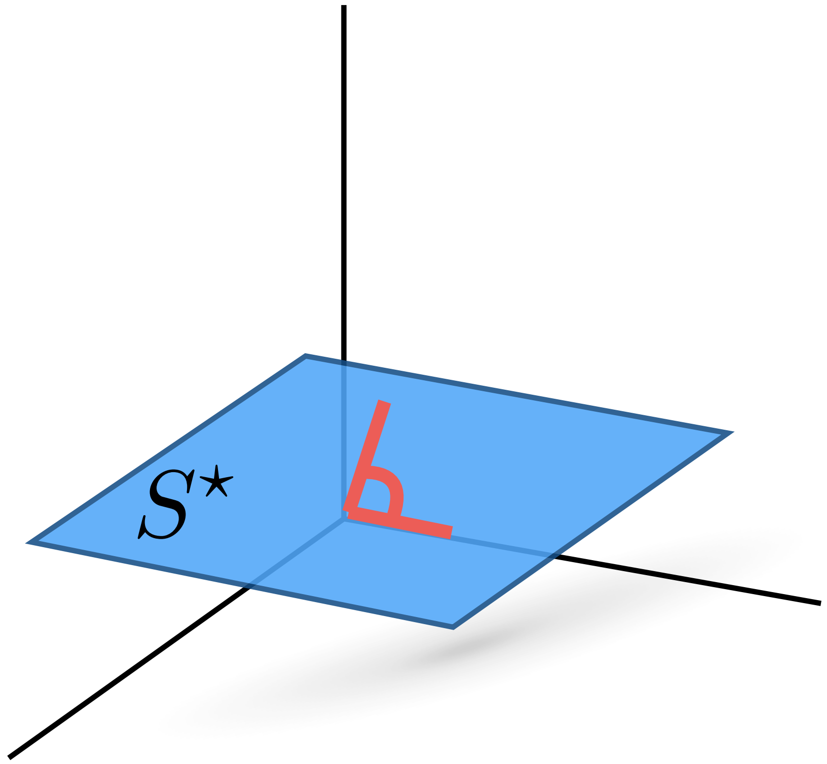
[10] Pimentel, Boston, Nowak, 2016

$S^\star = r$-dimensional subspace of $\mathbb{R}^d$ (in general position).



More formally

$S^\star = r$-dimensional subspace of $\mathbb{R}^d$ (in general position).



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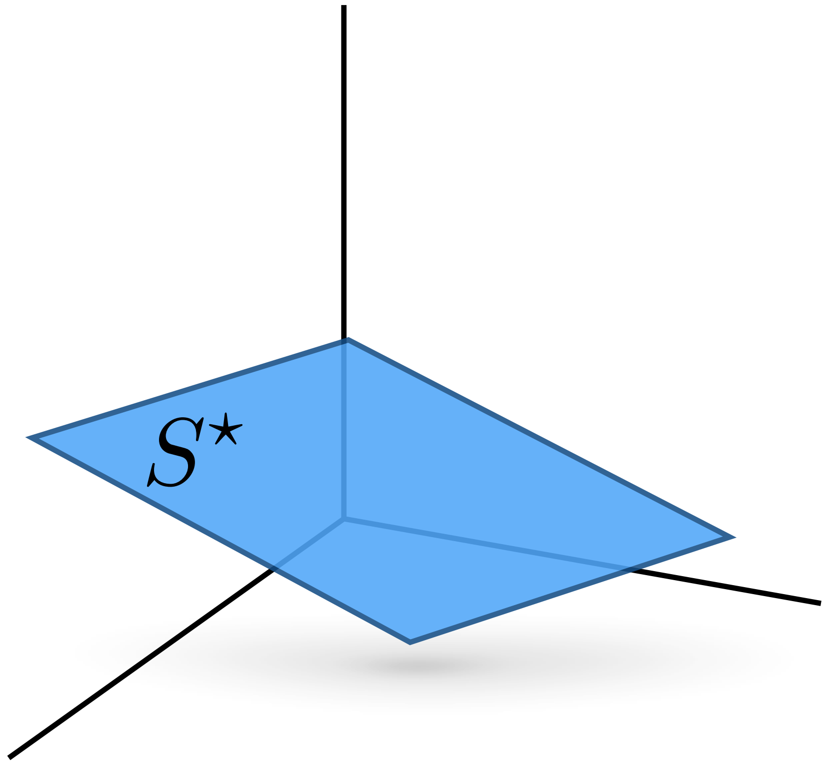
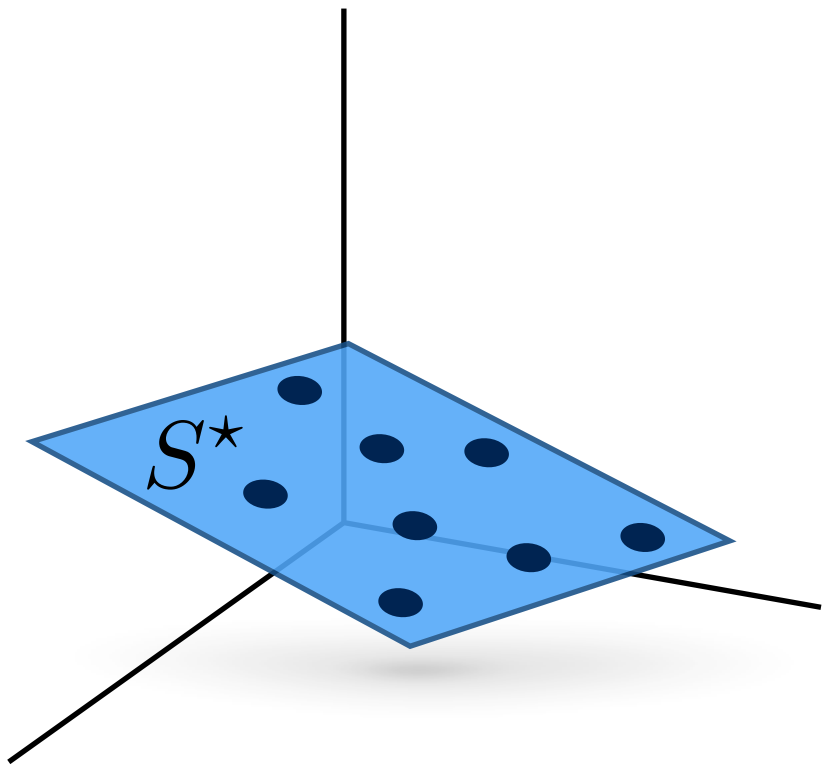


More formally

$S^\star$ = r-dimensional subspace of $\mathbb{R}^d$ (in general position).

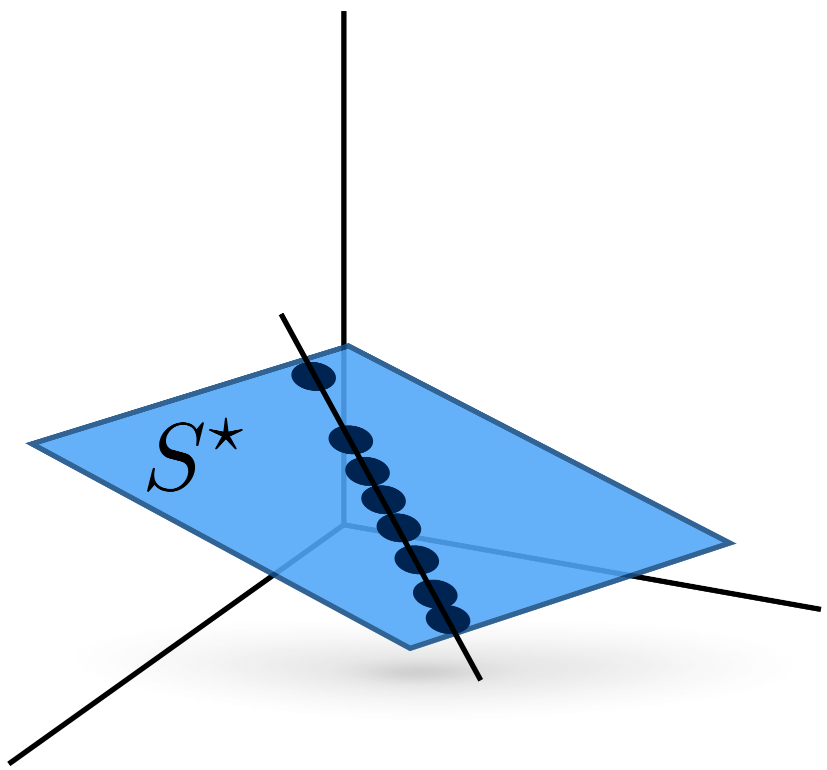


$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 3 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 3 & 3 & 6 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 3 & 3 & 9 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 6 & 6 & 9 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 6 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 6 & 6 & 9 & 4 & 3 & 6 & 2 \end{bmatrix}$$

Columns of $\mathbf{X}$ lie in $S^\star$ (generically).

More formally

$S^\star = r$-dimensional subspace of $\mathbb{R}^d$ (in general position).



$$\mathbf{X} \quad = \begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 3 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 3 & 3 & 6 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 3 & 3 & 9 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 6 & 6 & 9 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 6 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 6 & 6 & 9 & 4 & 3 & 6 & 2 \end{bmatrix}$$

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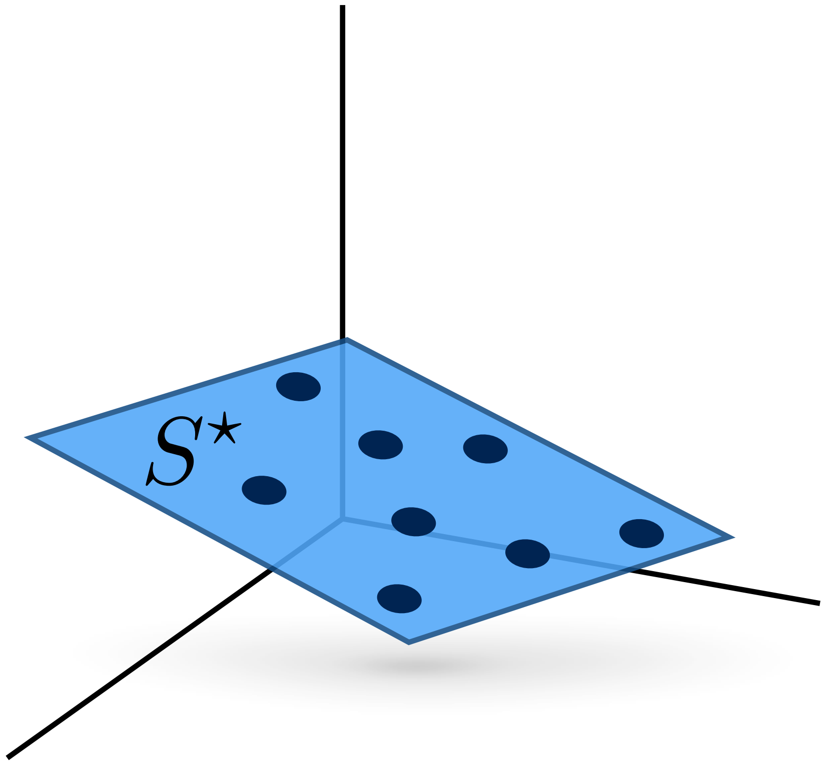
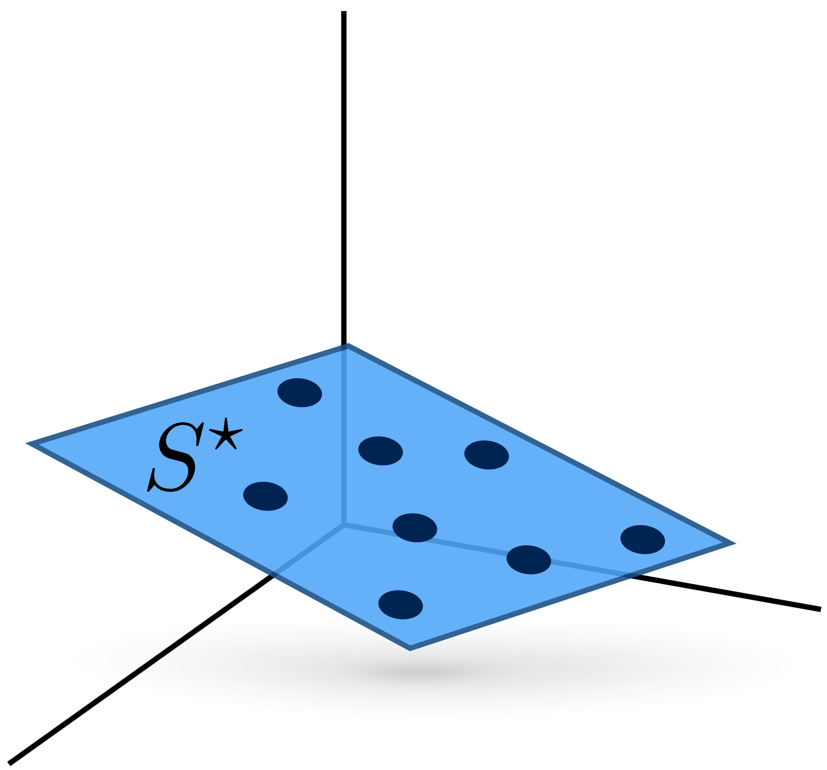


$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 3 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 3 & 3 & 6 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 3 & 3 & 9 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 6 & 6 & 9 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 6 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 6 & 6 & 9 & 4 & 3 & 6 & 2 \end{bmatrix}$$

Columns of $\mathbf{X}$ lie in $S^\star$ (generically).

More formally

$S^{\star} = r$-dimensional subspace of $\mathbb{R}^d$ (in general position).

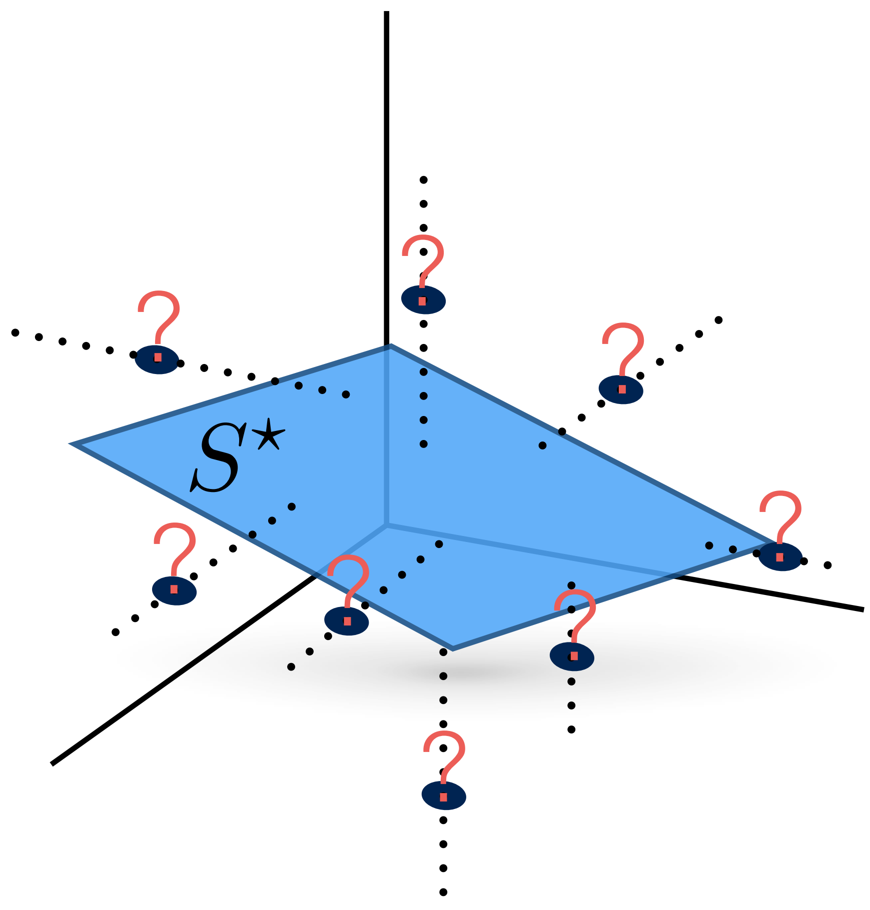


$$\mathbf{X} = \begin{bmatrix} 1 & & & 3 & & 3 & & 1 & 2 & \\ 2 & & 2 & & & 6 & & & 4 & \\ & & 3 & 3 & & 9 & & 3 & 6 & \\ 1 & & 1 & & 6 & & 4 & 1 & 2 & 2 \\ & 8 & & & 6 & & 4 & & & \\ & 8 & & & & & 4 & & & 2 \end{bmatrix}$$

Columns of $\mathbf{X}$ lie in $S^{\star}$ (generically).

More formally

$S^\star = r$-dimensional subspace of $\mathbb{R}^d$ (in general position).



$$\mathbf{X} = \begin{bmatrix} 1 & & & 3 & & 3 & & 1 & 2 & \\ 2 & & 2 & & & 6 & & & 4 & \\ & & 3 & 3 & & 9 & & 3 & 6 & \\ 1 & & 1 & & 6 & & 4 & 1 & 2 & 2 \\ & 8 & & & 6 & & 4 & & & \\ & 8 & & & & & 4 & & & 2 \end{bmatrix}$$

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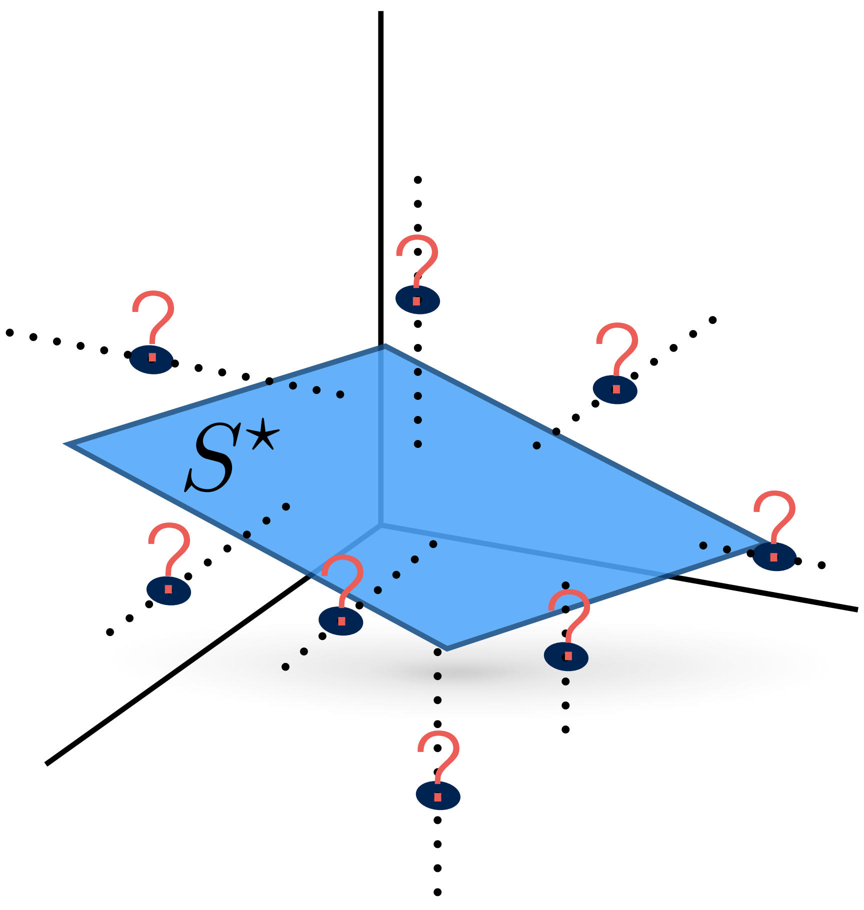


$$\mathbf{X} = \begin{bmatrix} 1 & & & 3 & & 3 & & 1 & 2 & \\ 2 & & 2 & & & 6 & & & 4 & \\ & & 3 & 3 & & 9 & & 3 & 6 & \\ 1 & & 1 & & 6 & & 4 & 1 & 2 & 2 \\ & 8 & & & 6 & & 4 & & & \\ & 8 & & & & & 4 & & & 2 \end{bmatrix}$$

Columns of $\mathbf{X}$ lie in $S^\star$ (generically).

GOAL: Recover $S^\star$.

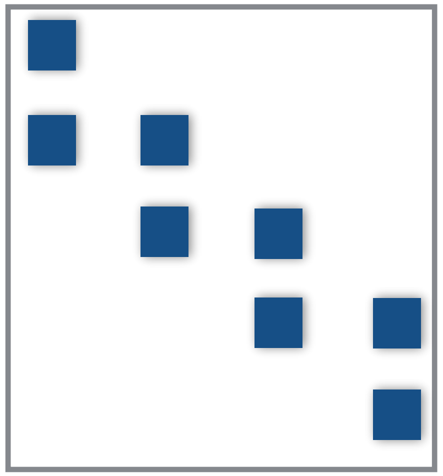
More formally

Let $\mathbf{X}_\tau$ be a matrix formed with $d - r$ columns of $\mathbf{X}$.
We say $\mathbf{X}_\tau$ is *observed in the right entries* if every subset of n columns of $\mathbf{X}_\tau$ has observations on at least $n + r$ rows.

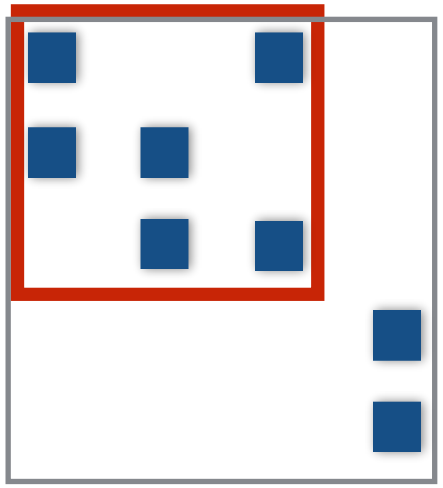
Observed in the right entries

What do I mean?

Let $\mathbf{X}_\tau$ be a matrix formed with $d-r$ columns of $\mathbf{X}$.
We say $\mathbf{X}_\tau$ is *observed in the right entries* if every subset of n columns of $\mathbf{X}_\tau$ has observations on at least $n+r$ rows.

$\mathbf{X}_\tau =$ 

Good

$\mathbf{X}_\tau =$ 

Bad

Observed in the right entries

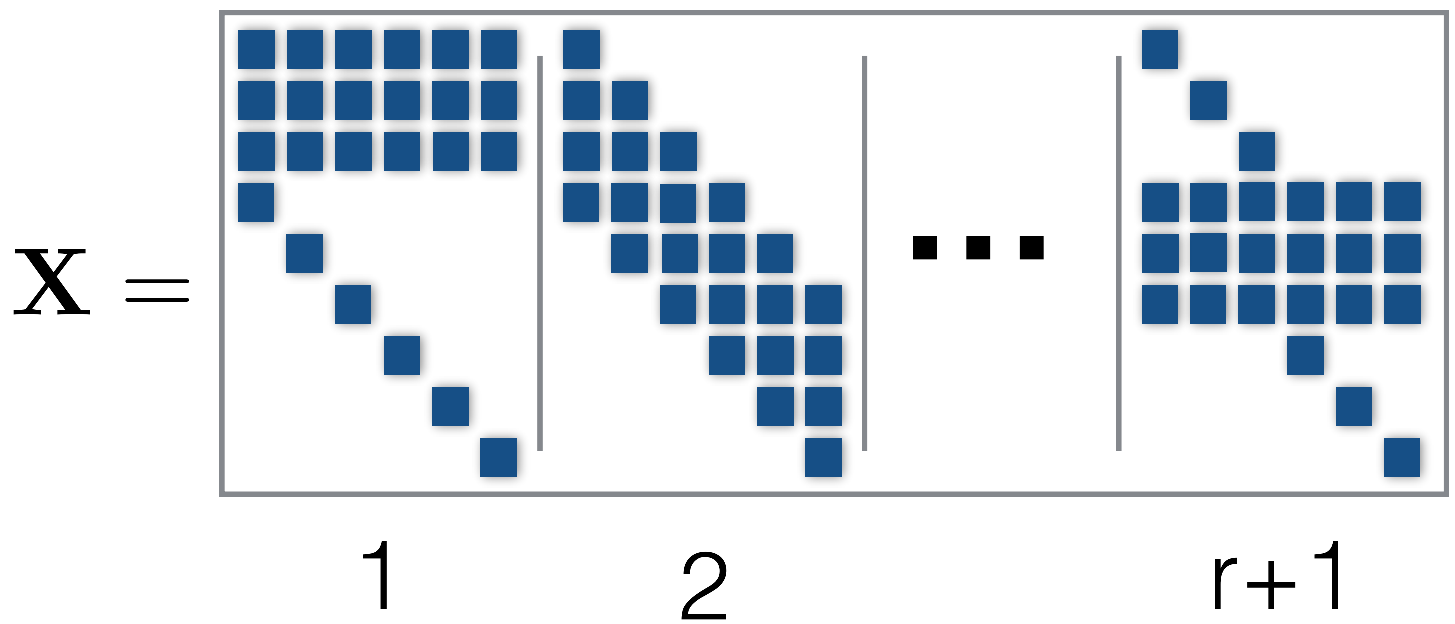
What do I mean?

Suppose $\mathbf{X}$ contains $r+1$ disjoint matrices $\{\mathbf{X}_\tau\}_{\tau=1}^{r+1}$ *observed in the right entries*. Then $S^\star$ is the only r-dimensional subspace that agrees with $\mathbf{X}$.

Our Main Result

[10] Pimentel, Boston, Nowak, 2016

Suppose $\mathbf{X}$ contains $r+1$ disjoint matrices $\{\mathbf{X}_\tau\}_{\tau=1}^{r+1}$ *observed in the right entries*. Then $S^\star$ is the only r-dimensional subspace that agrees with $\mathbf{X}$.



Our Main Result

[10] Pimentel, Boston, Nowak, 2016

Our main result in a nutshell

[10] Pimentel, Boston, Nowak, 2016

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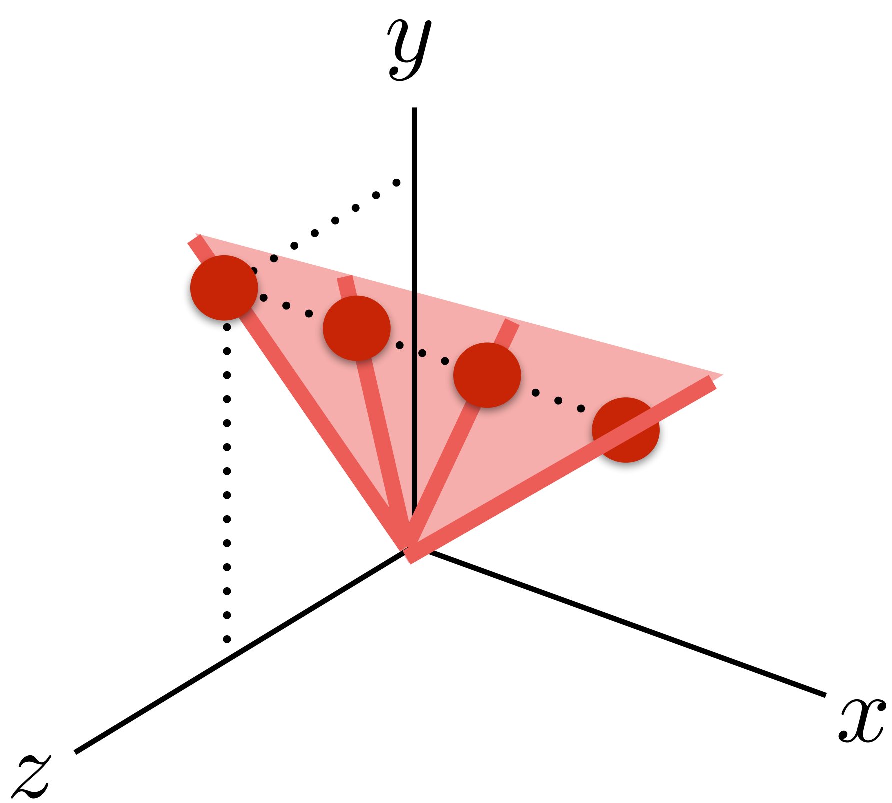
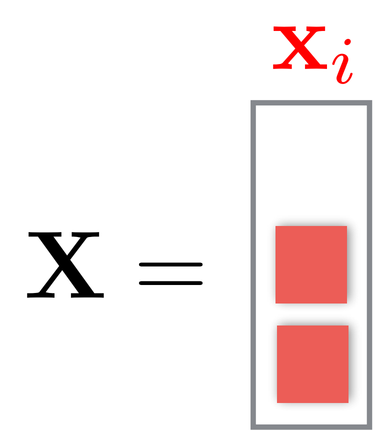
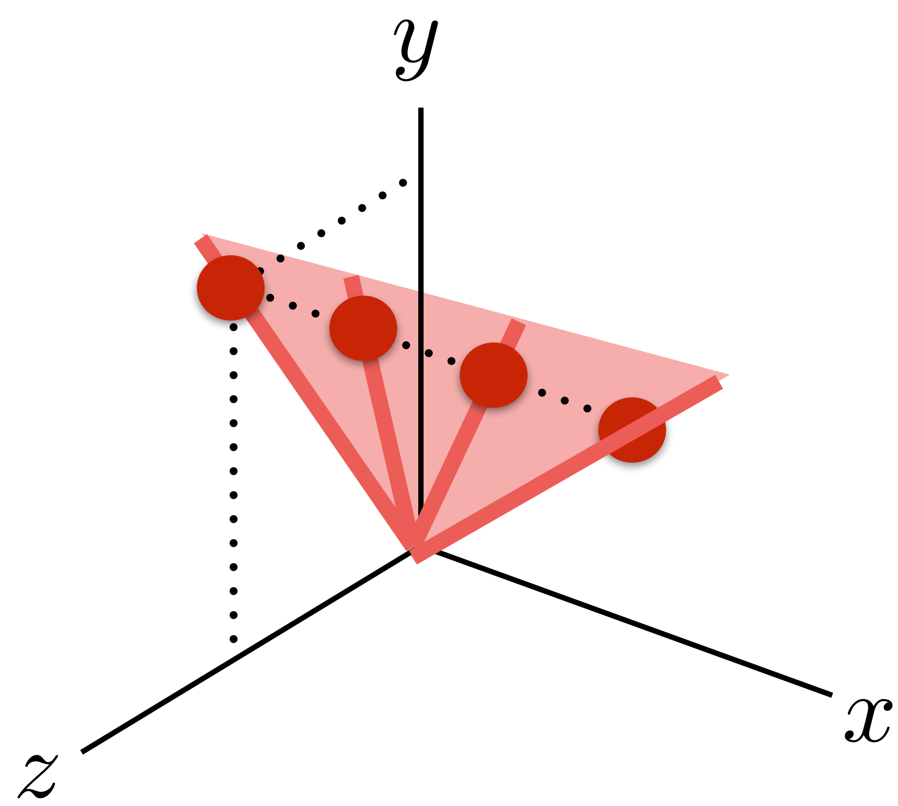
This column imposes *1* restriction on what the subspace may be

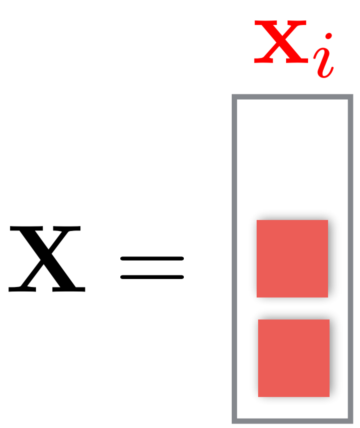




Main idea of the proof

Each column with *r+1* entries imposes
1 restriction on what the subspace may be

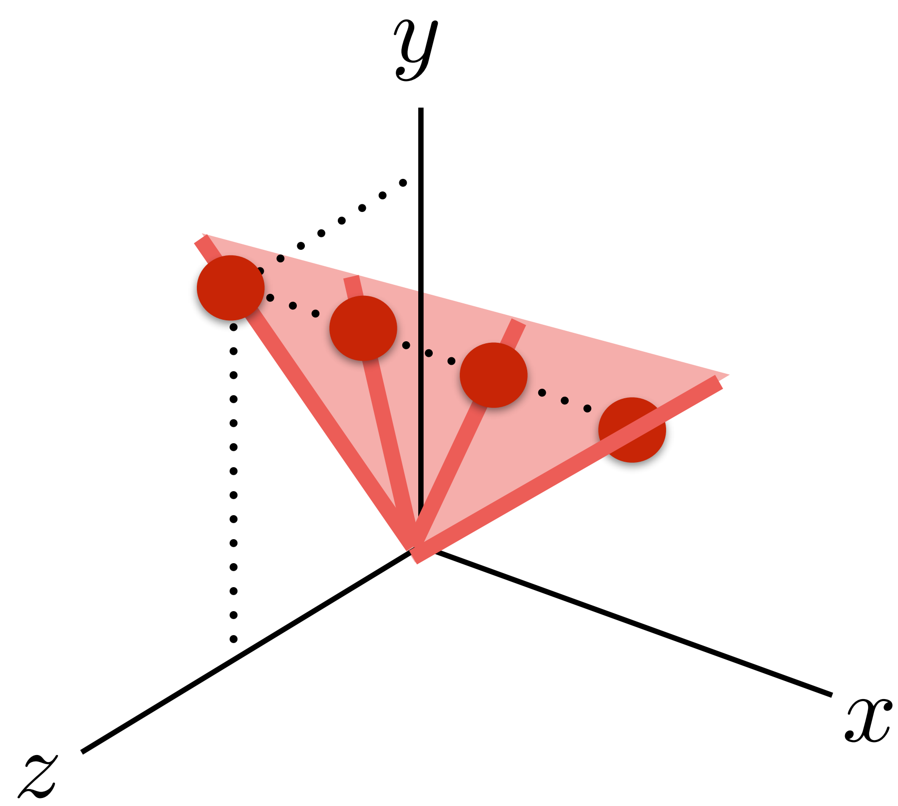


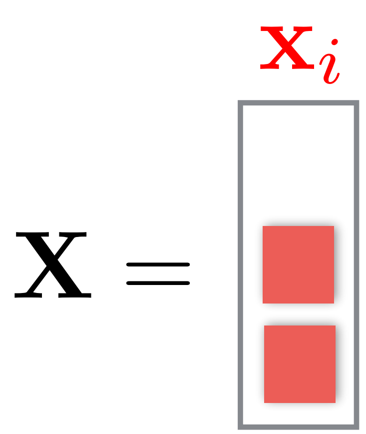


Main idea of the proof

Each column with $r+1$ entries imposes 1 restriction on what the subspace may be

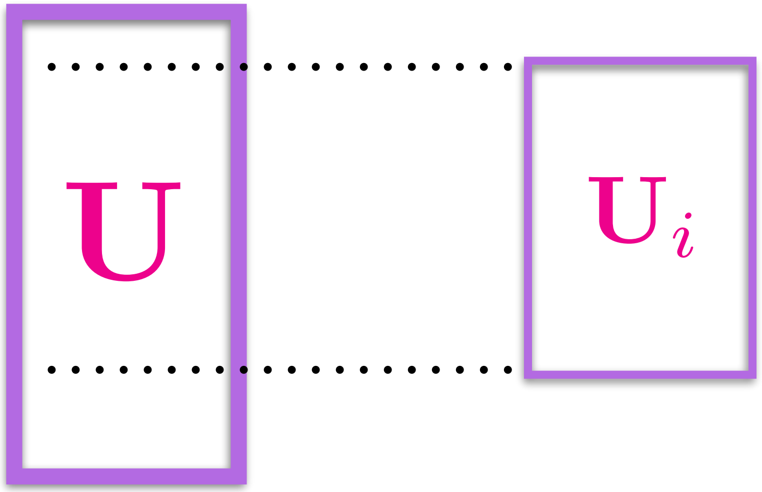
Degree-r polynomial





Main idea of the proof

Take a basis of an arbitrary subspace



This subspace agrees with $\mathbf{x}_i$ if and only if

$$\mathbf{x}_i = \mathbf{U}_i \boldsymbol{\theta}_i$$

Main idea of the proof

- We can split this as:

$$r\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \!\!\!\! 1\,\{ \quad \left[\begin{array}{c} \boldsymbol{x}_{\boldsymbol{\Delta}_i} \\ \hline \boldsymbol{x}_{\boldsymbol{\nabla}_i} \end{array}\right] = \left[\begin{array}{c} \mathbf{U}_{\boldsymbol{\Delta}_i} \\ \hline \mathbf{U}_{\boldsymbol{\nabla}_i} \end{array}\right] \boldsymbol{\theta}_i.$$

- We can use the top block to solve for $\boldsymbol{\theta}_i$:

$$\boldsymbol{\theta}_i = \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}.$$

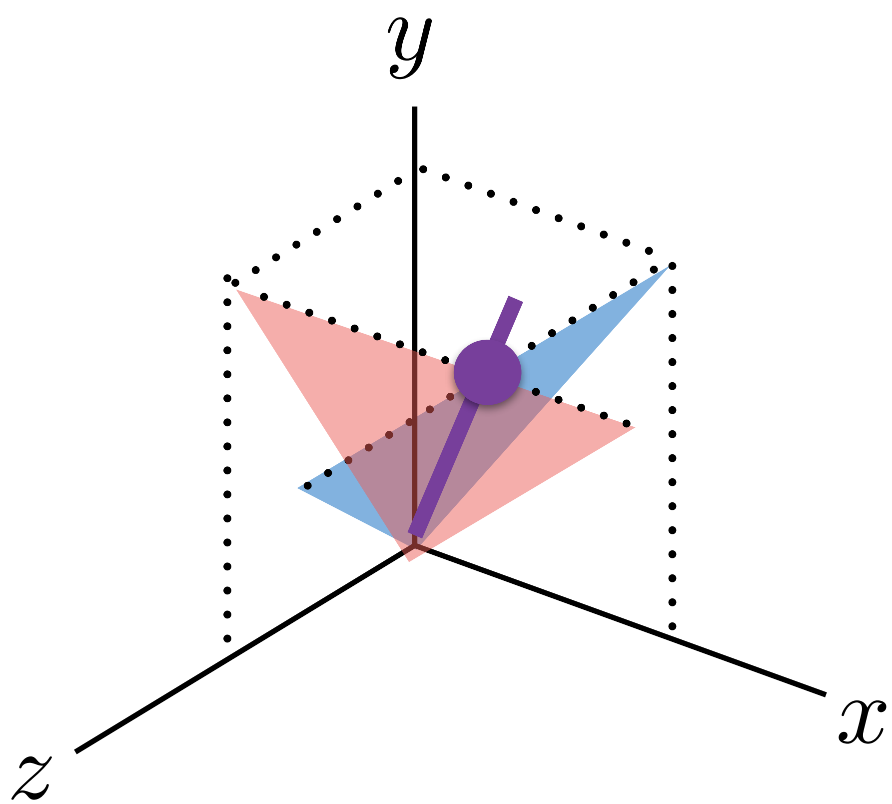
- Plug this in the last row:

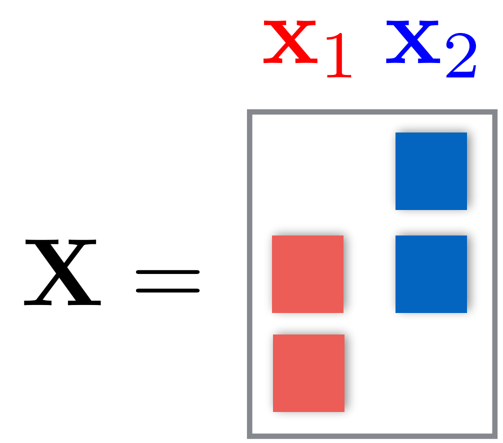
$$\boldsymbol{x}_{\boldsymbol{\nabla}_i} = \mathbf{U}_{\boldsymbol{\nabla}_i} \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}.$$

- Or equivalently

$$\underbrace{\boldsymbol{x}_{\boldsymbol{\nabla}_i} - \mathbf{U}_{\boldsymbol{\nabla}_i} \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}}_{f_i(\mathbf{U}_i | x_i)} = 0.$$

Main idea of the proof



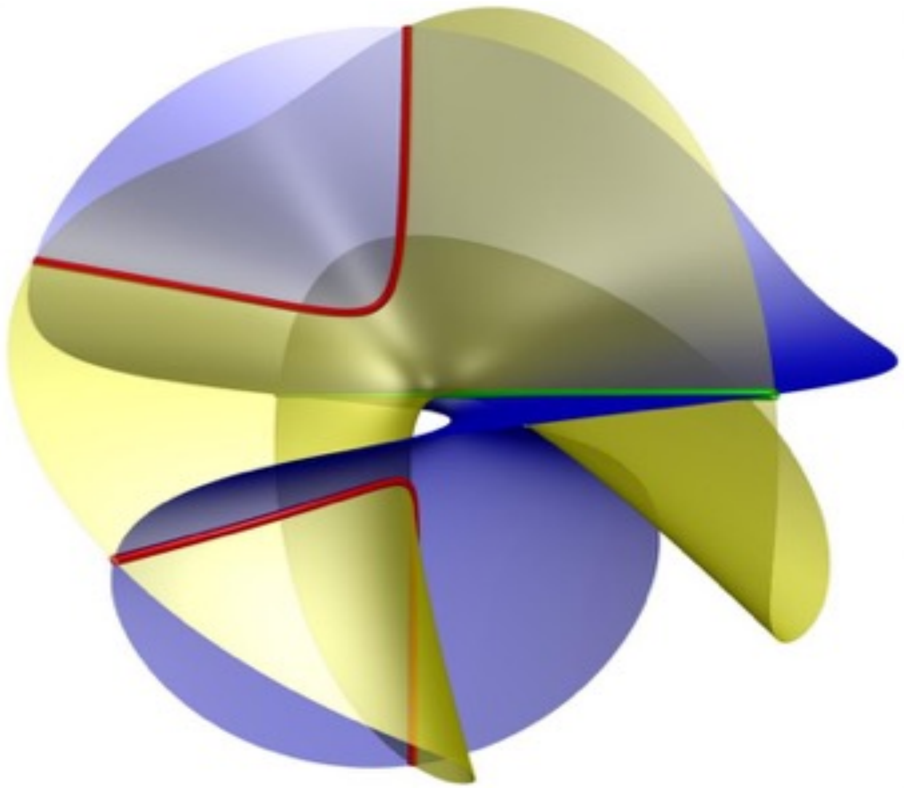


A subspace S agrees with $\mathbf{X}$

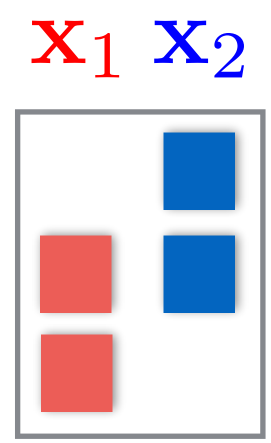
$$\Updownarrow$$

$$f_1(\mathbf{U}_1|x_1) = 0$$
$$f_2(\mathbf{U}_2|x_2) = 0$$

Main idea of the proof



$\mathbf{X} =$



A subspace S agrees with $\mathbf{X}$

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$$f_1(\mathbf{U}_1|x_1) = 0$$
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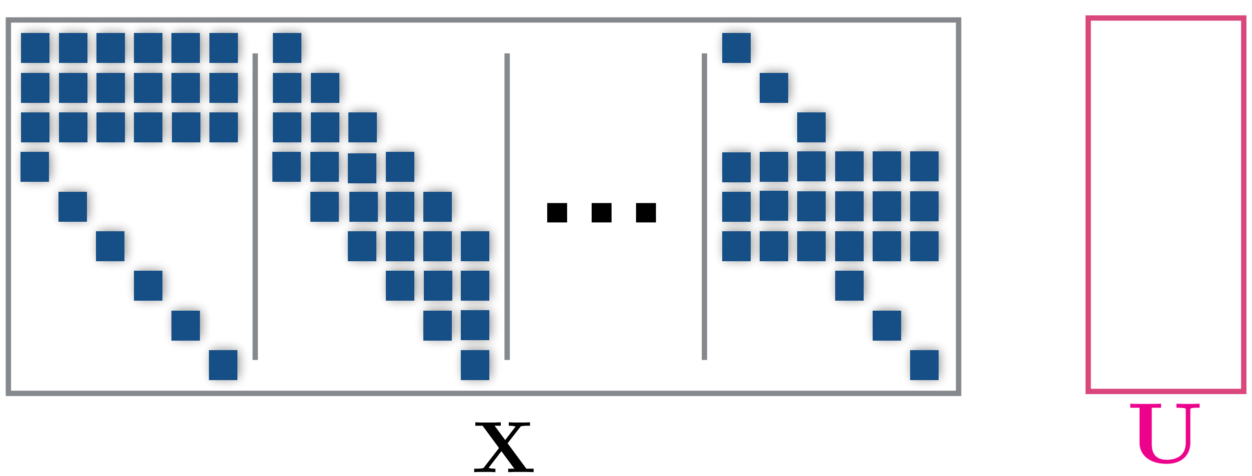
Main idea of the proof

- Each of column produces one polynomial

$$f_1(\mathbf{U}_1|x_1),\ f_2(\mathbf{U}_2|x_2),\ \ldots,\ f_N(\mathbf{U}_N|x_N)$$

- The observed rows indicate the variables involved
- If data is observed in *the right entries*, all variables will be pined down



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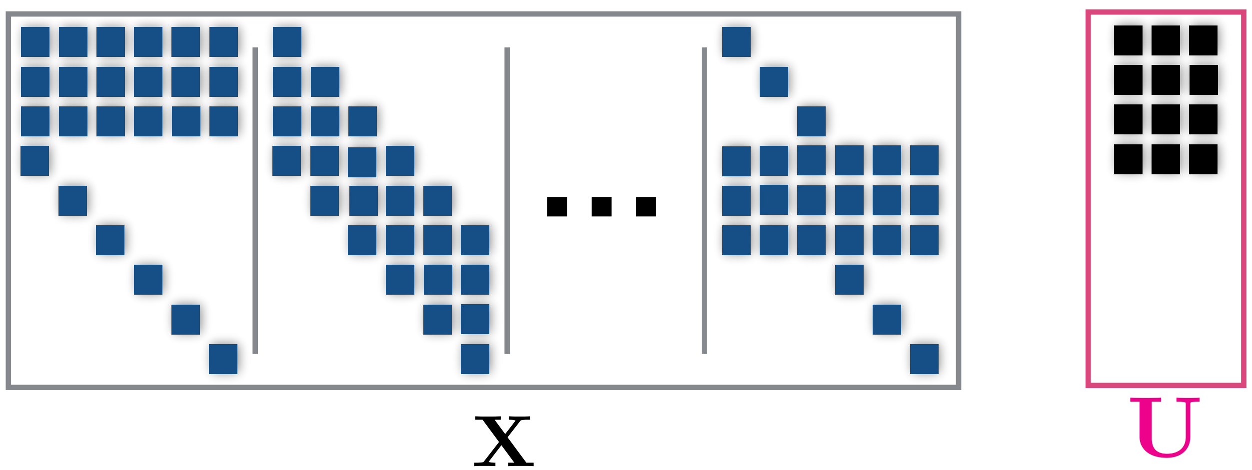

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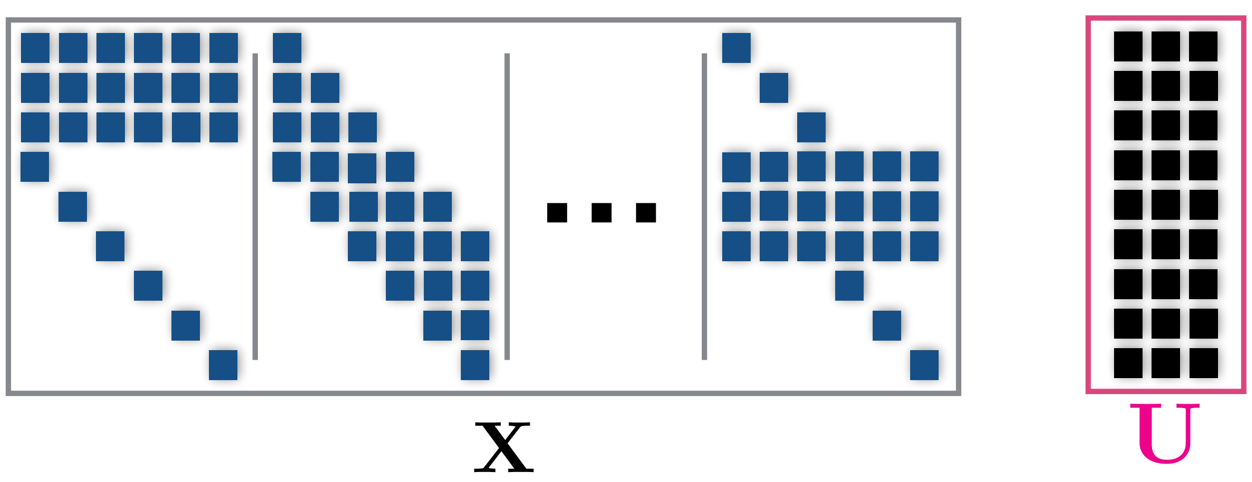

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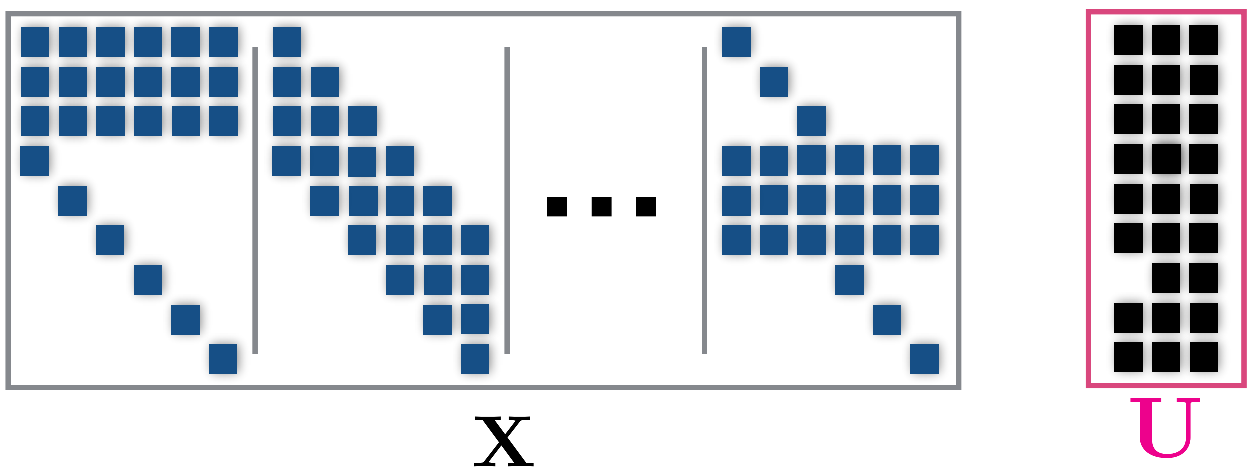

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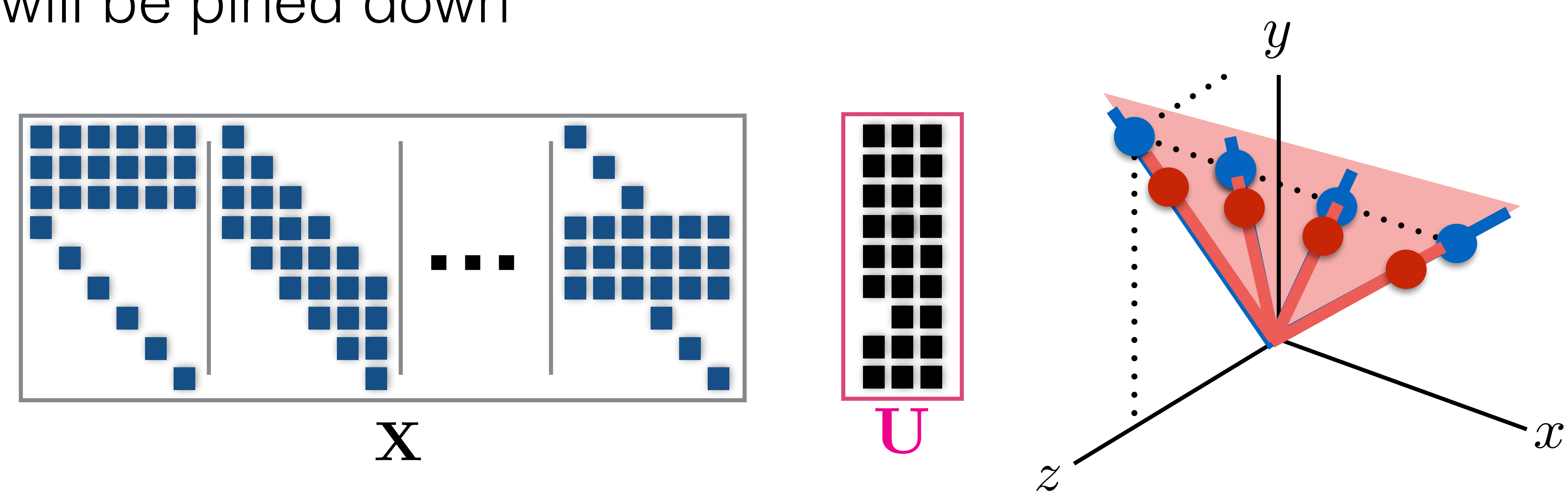


Main idea of the proof

- If data is observed in *the right entries*
  - Polynomials are algebraically independent

- After this, use cool Algebraic Geometry tricks:
  - Polynomials are a regular sequence
  - Polynomials define a zero-dimensional variety
  - At most finitely many solutions
  - Unique solution (with a bit more work)

Main idea of the proof


What is this good for?

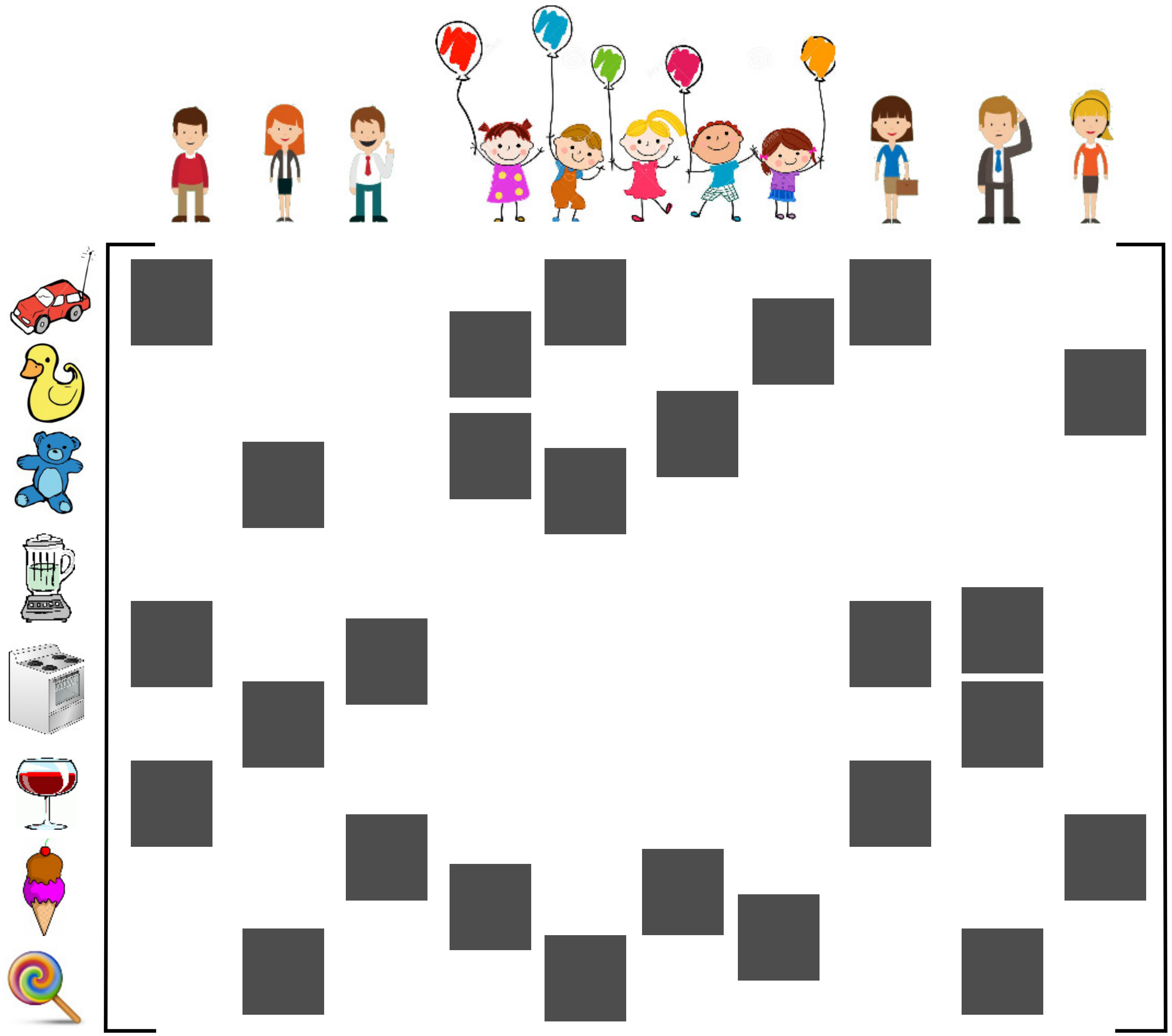

What is this good for?


OMG, OMG, OMG!! Say something!

What is this good for?

OMG, OMG, OMG!! Say something!

[3] Robust PCA (2017)
[7] Unions of Subspaces (2016)
[8] A Converse to MC (2016)
[9] Sampling Regimes (2016)
[10] Coherence (2016)
[10] Computational Complexity (2016)
[11] Adaptive Sampling (2015)
[12] Lower Bound (2015)
[13] Validation Criteria (2015)



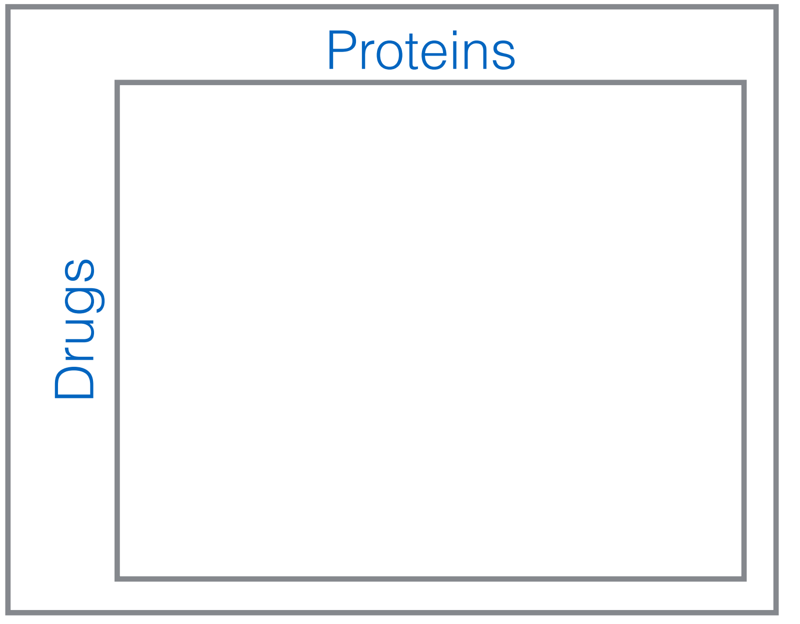
Adaptive Sampling

If we can choose, let's choose *the right entries!*

[11] Pimentel et. al, 2015

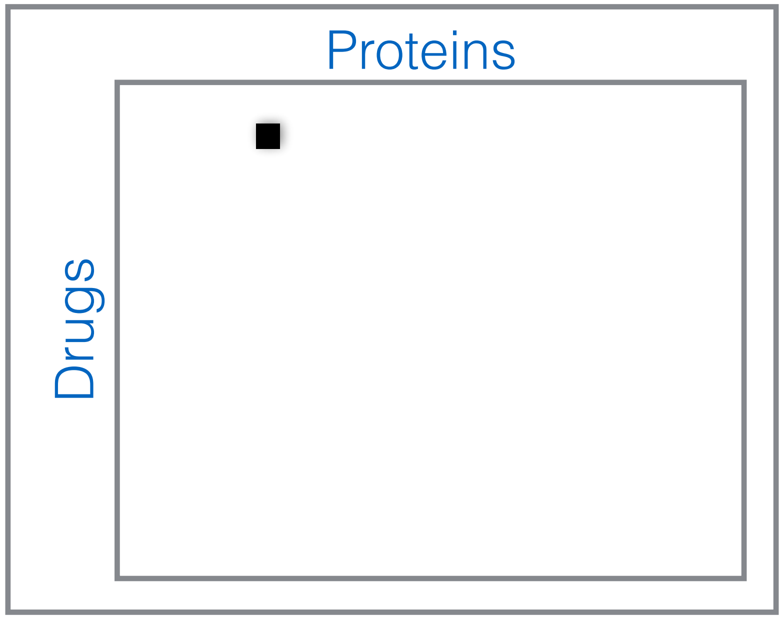
Drug Discovery

Adaptive Sampling



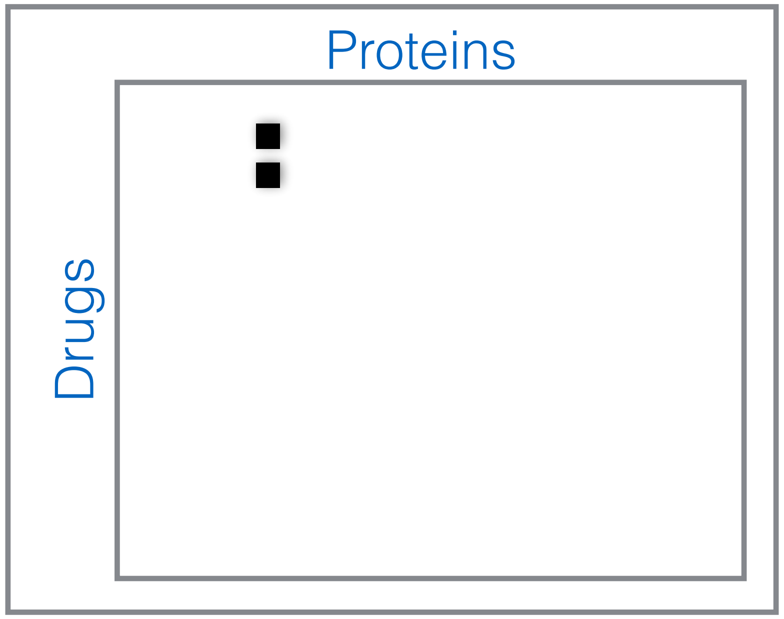
Drug Discovery

Adaptive Sampling



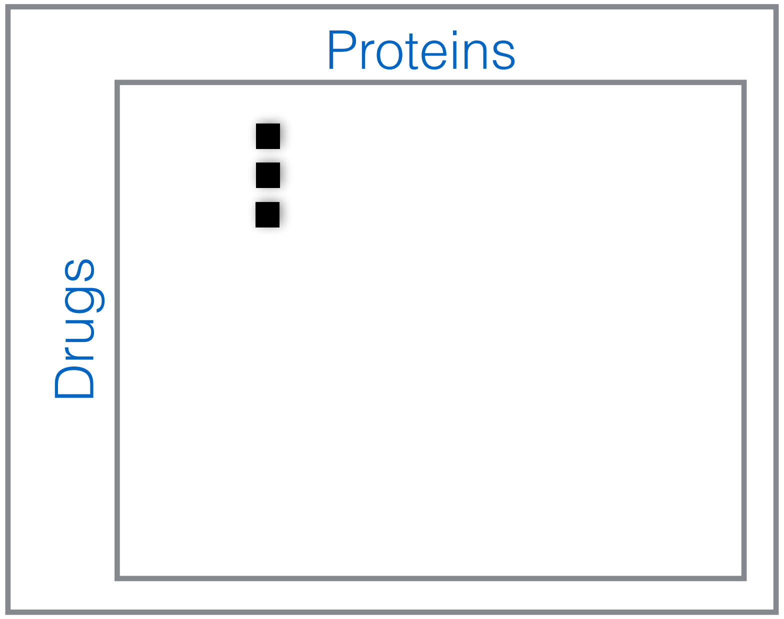
Drug Discovery

Adaptive Sampling



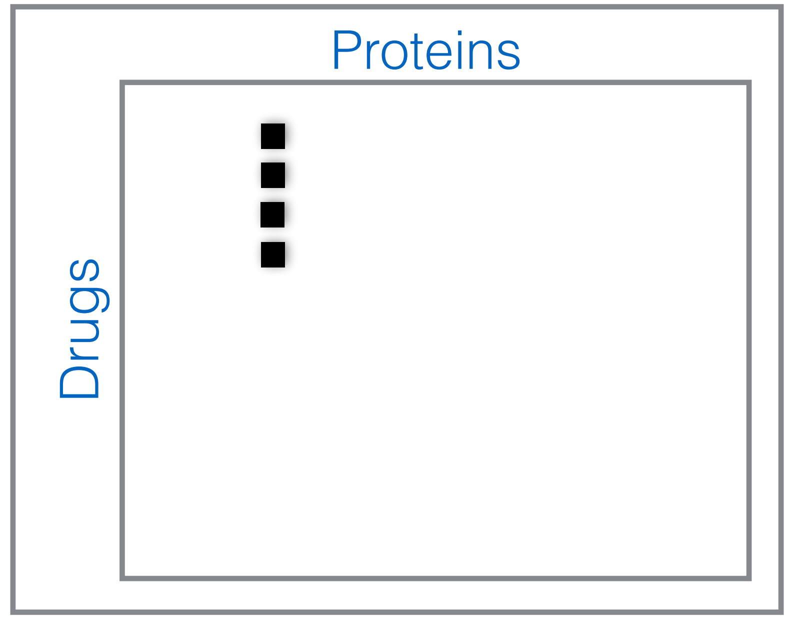
Drug Discovery

Adaptive Sampling



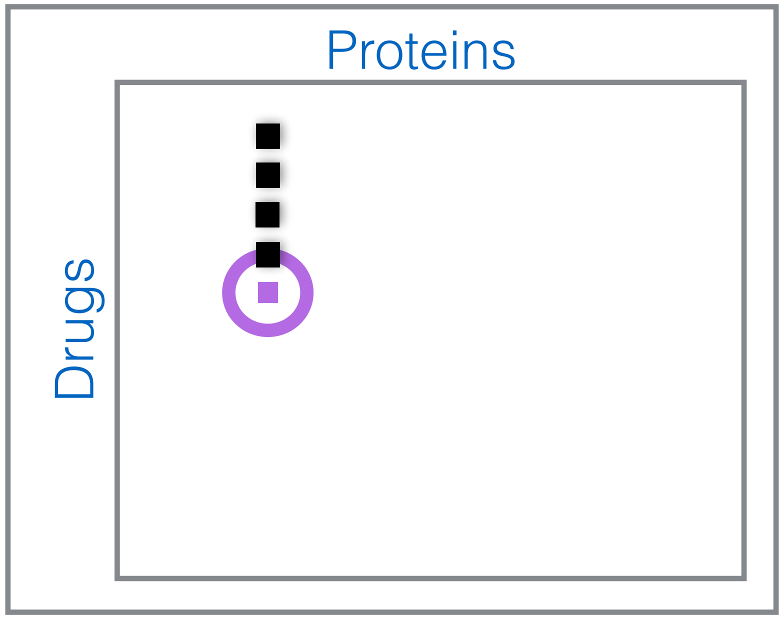
Drug Discovery

Adaptive Sampling



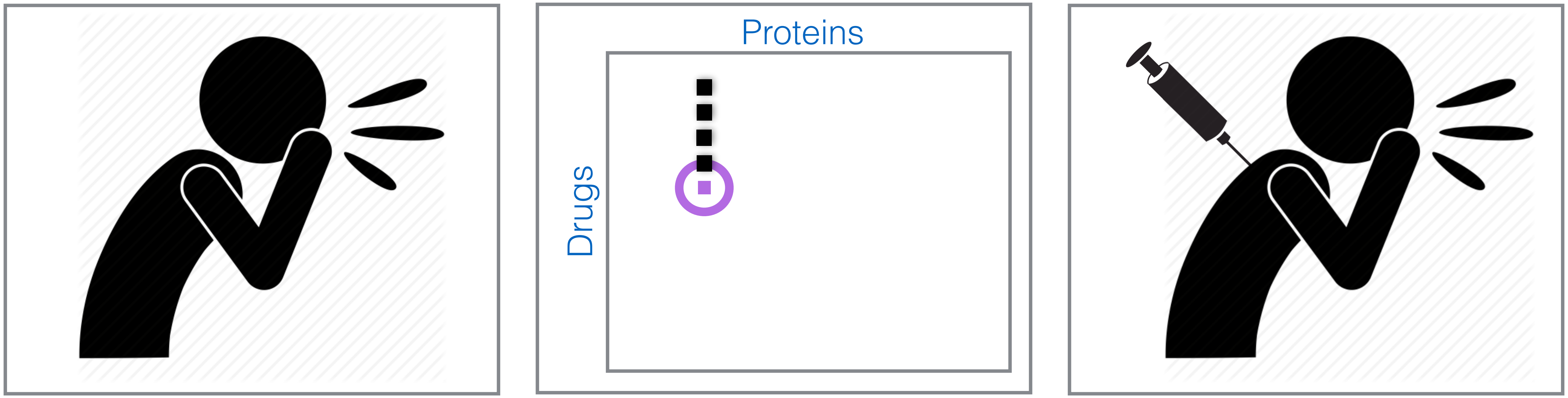
Drug Discovery

Adaptive Sampling



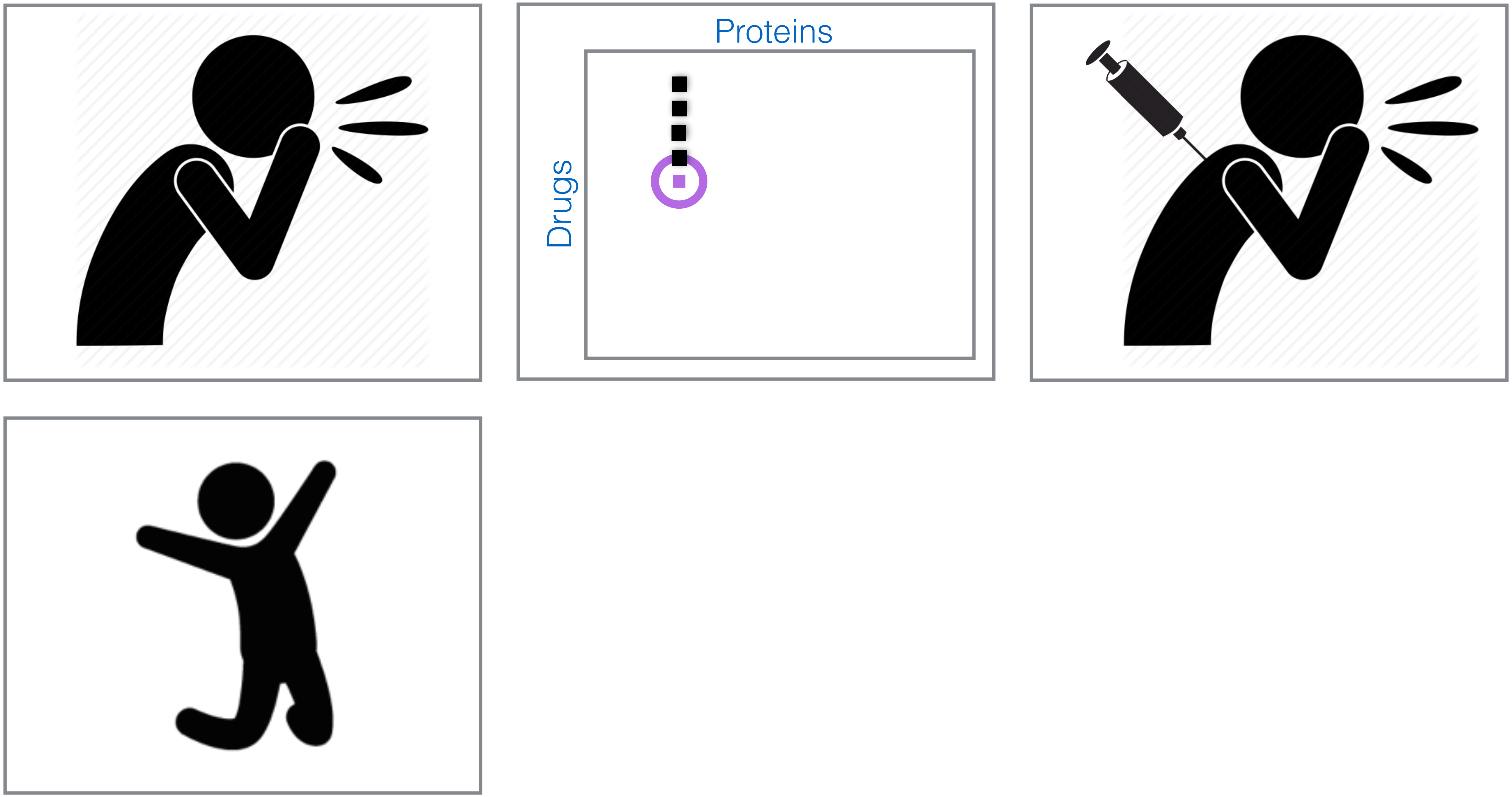
Drug Discovery

Adaptive Sampling



Drug Discovery

Adaptive Sampling



Drug Discovery

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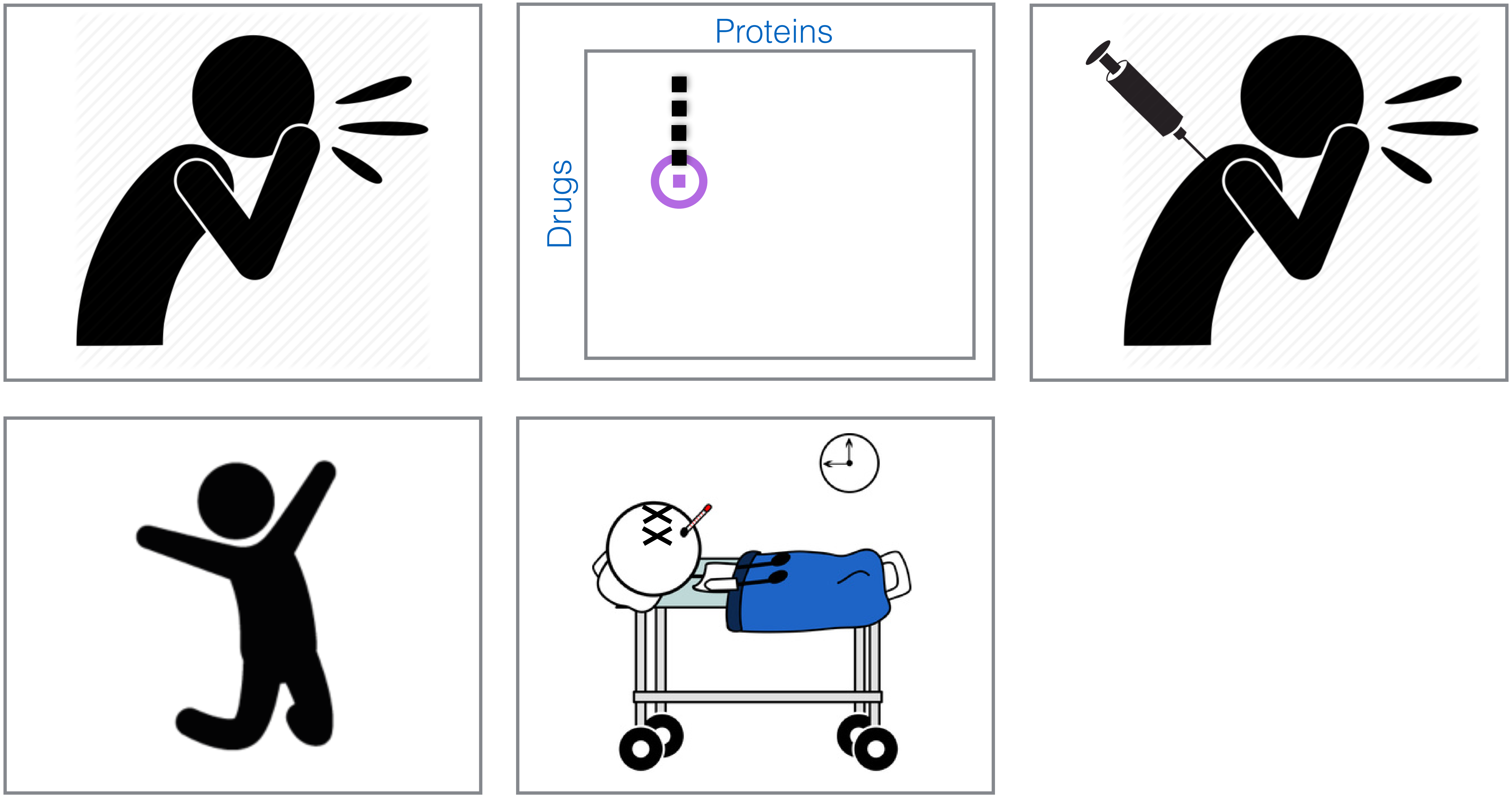

Drug Discovery

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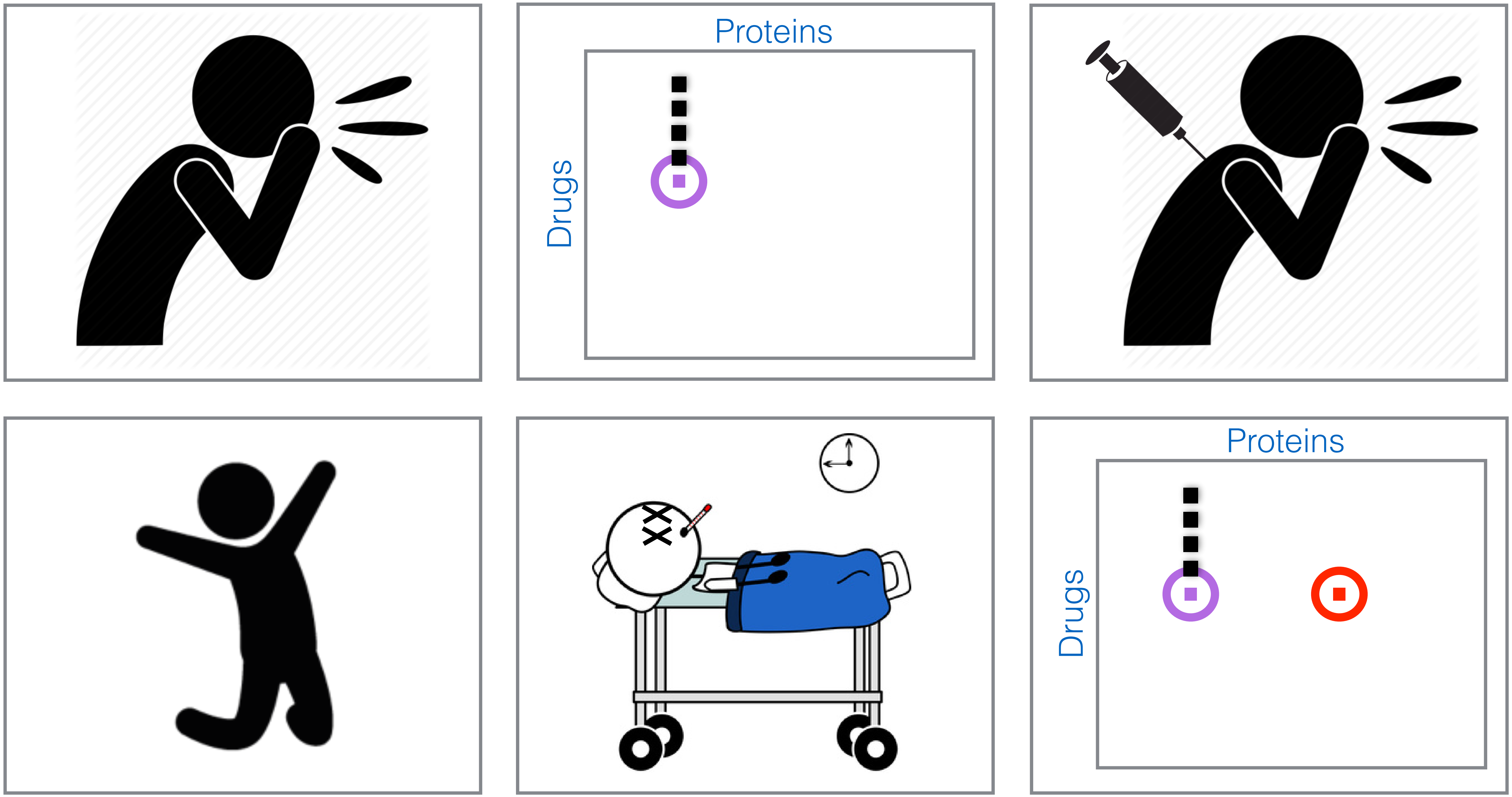
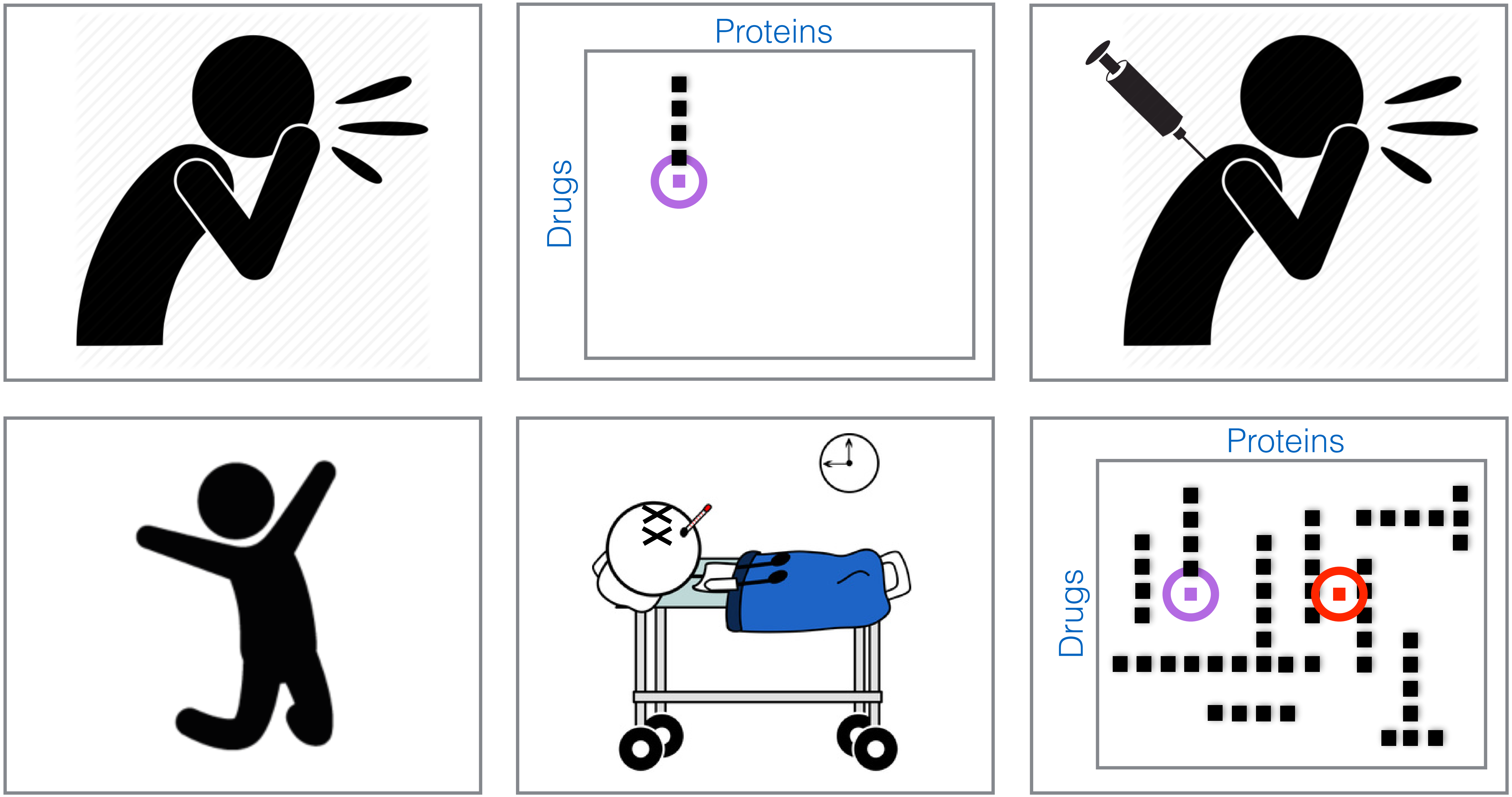


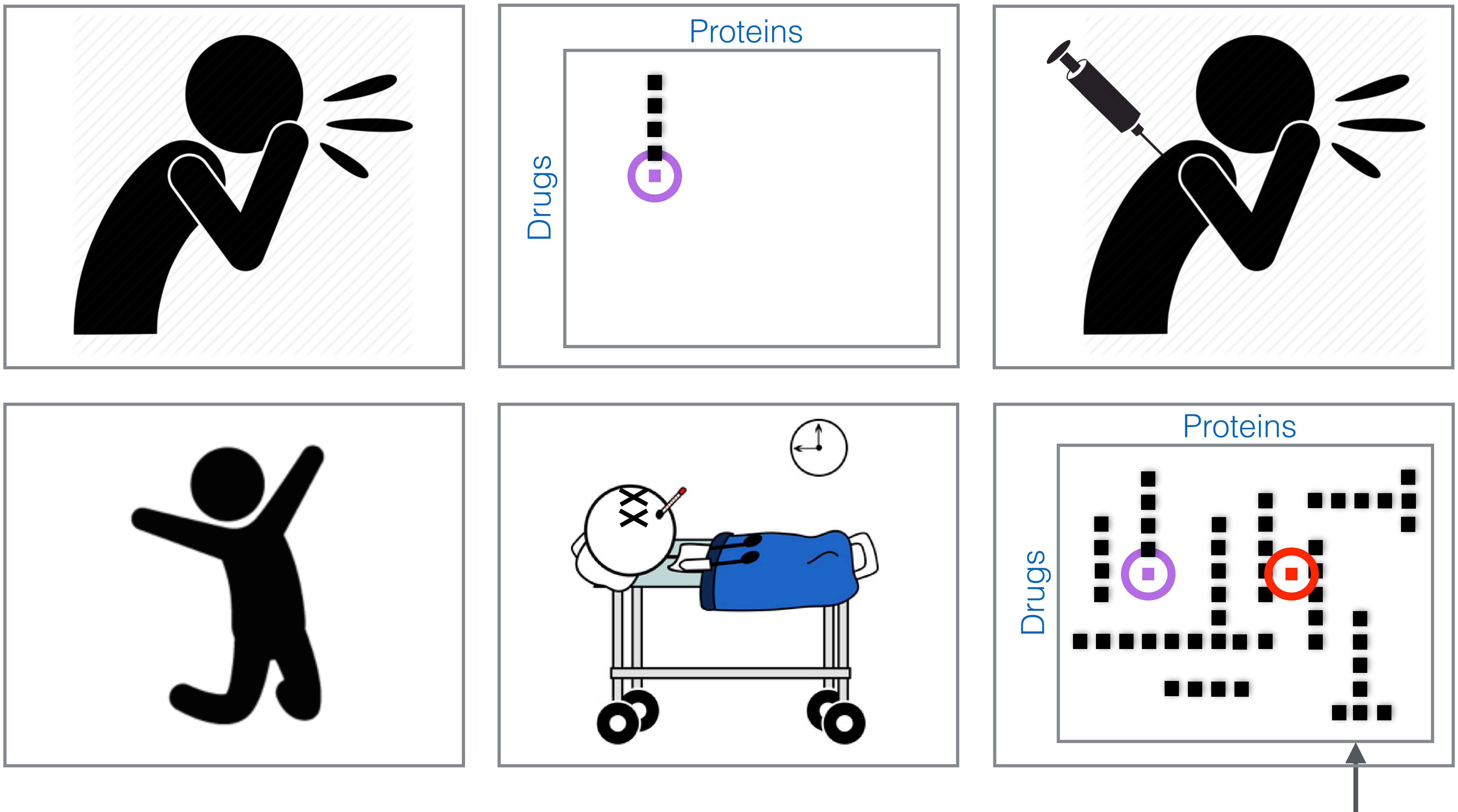
Drug Discovery

Adaptive Sampling



Drug Discovery

Adaptive Sampling



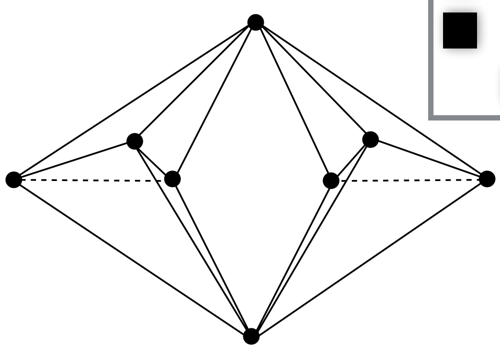
Columns in Subspace!

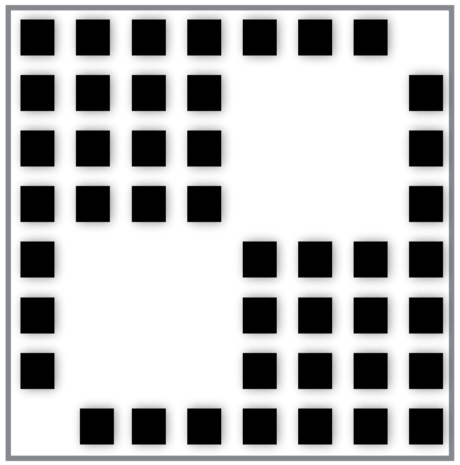
Drug Discovery

Adaptive Sampling

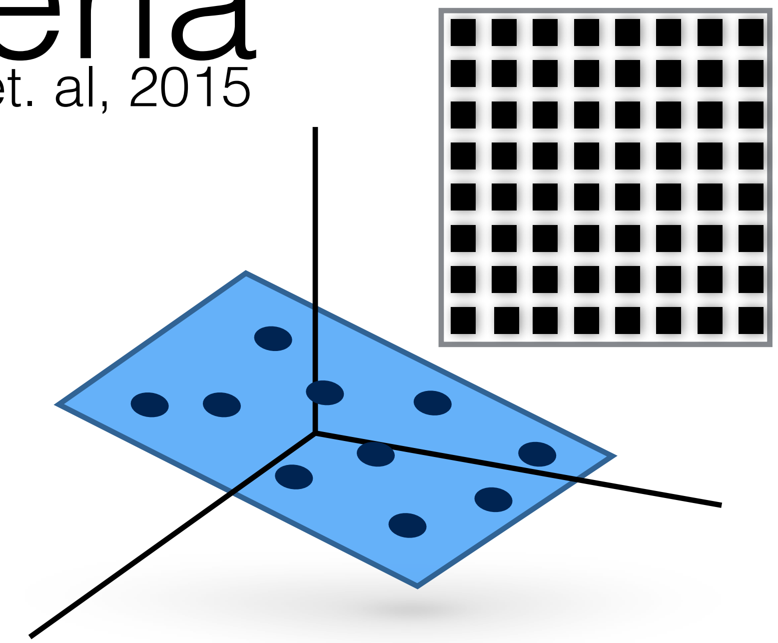
Validation Criteria

[13] Pimentel et. al, 2015





Any Algorithm!



Is this **the right** Subspace?

If data was observed in the right entries, YES.

- Regardless of coherence
- Arbitrary Sampling
- With probability 1

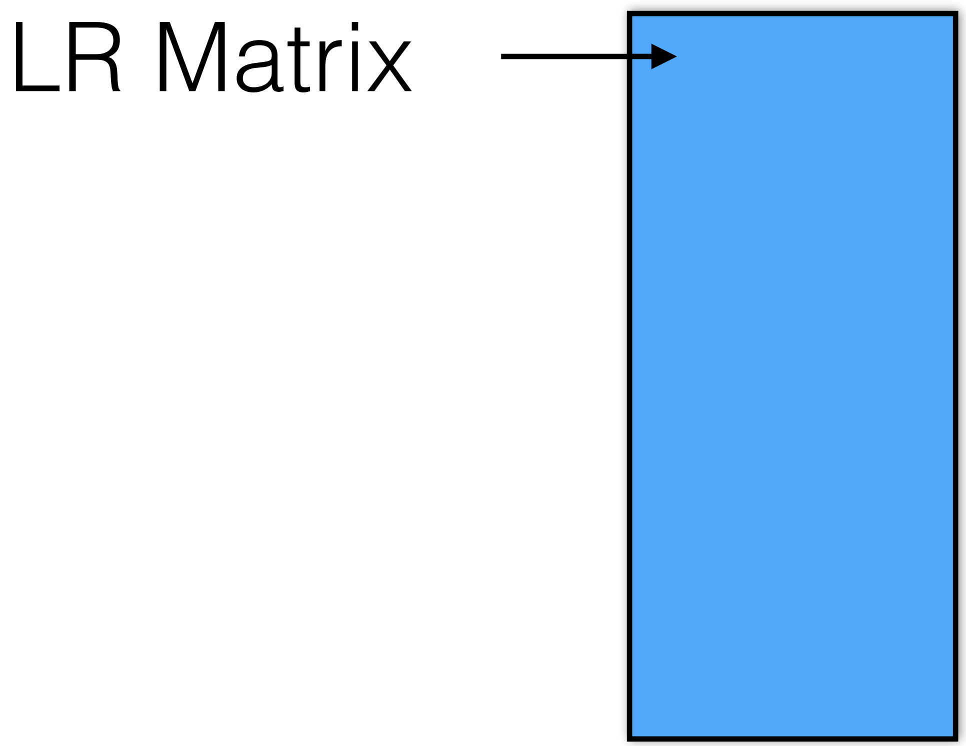

WOW, AMAZING
PLEASE TELL
ME MORE

OK, OK. I'll tell you about ONE more application:

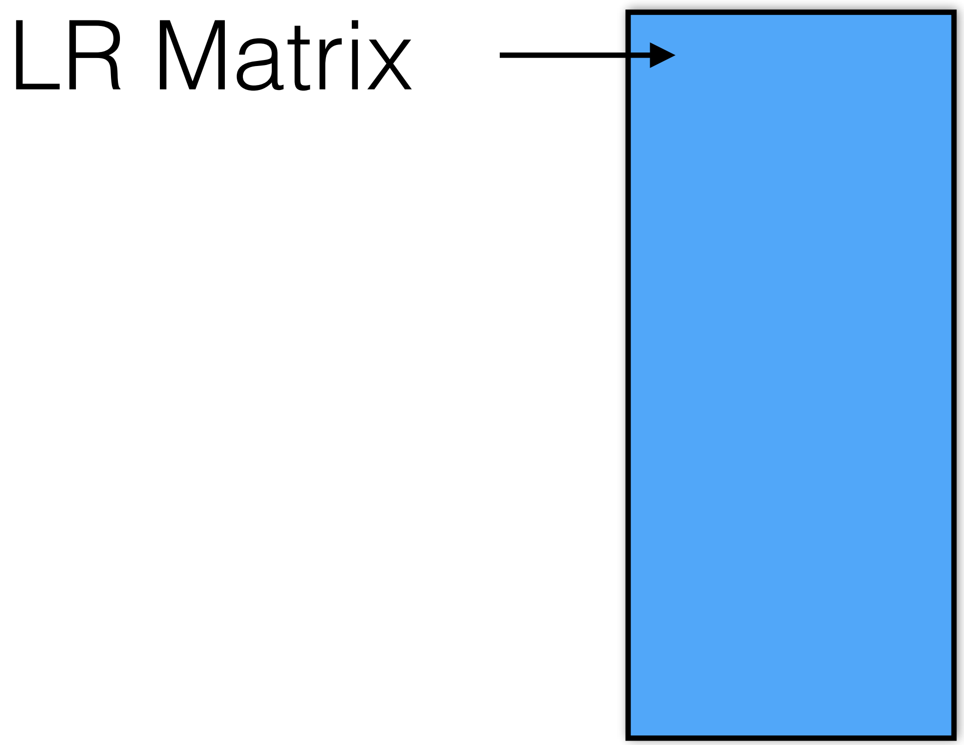
Robust PCA

In a completely different way

LRMC
(Low-Rank Matrix Completion)



RPCA
(Robust Principal Component Analysis)



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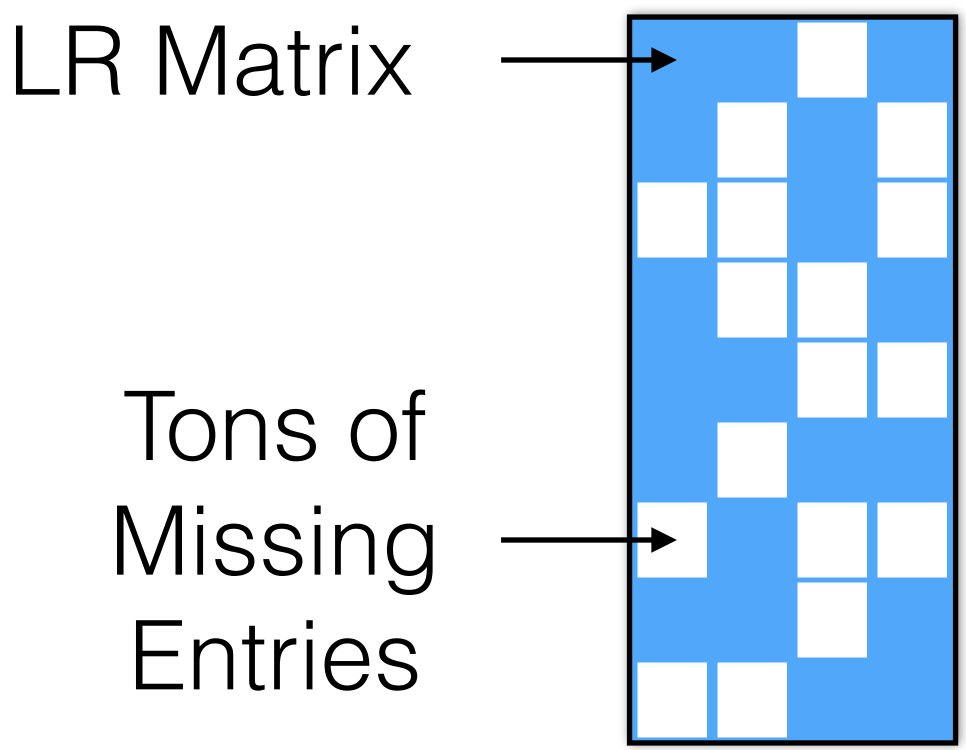

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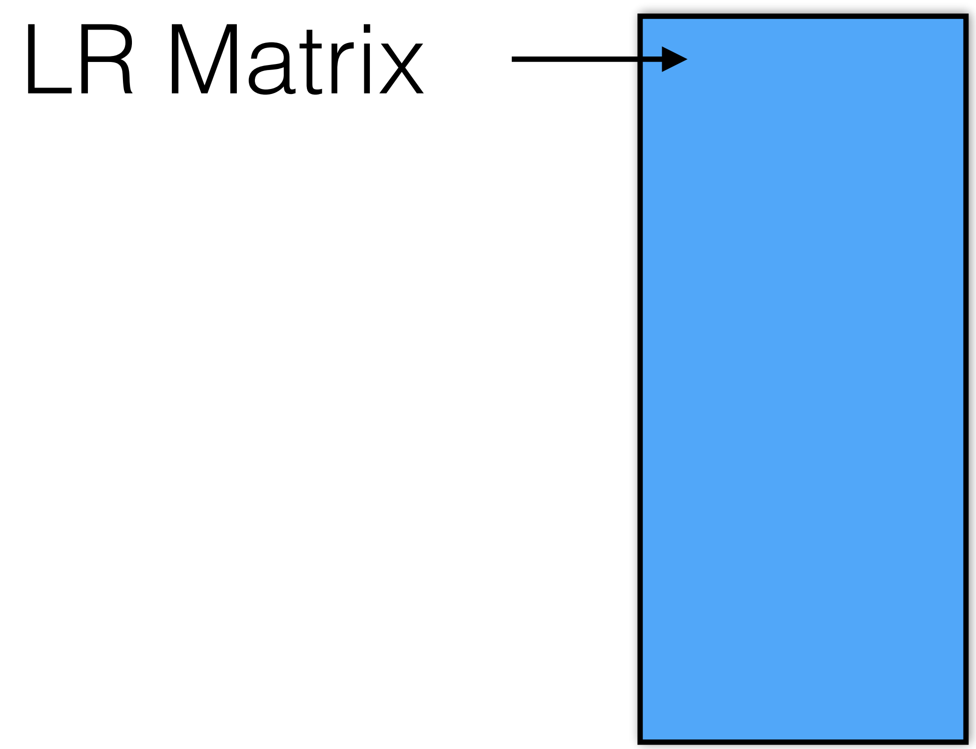

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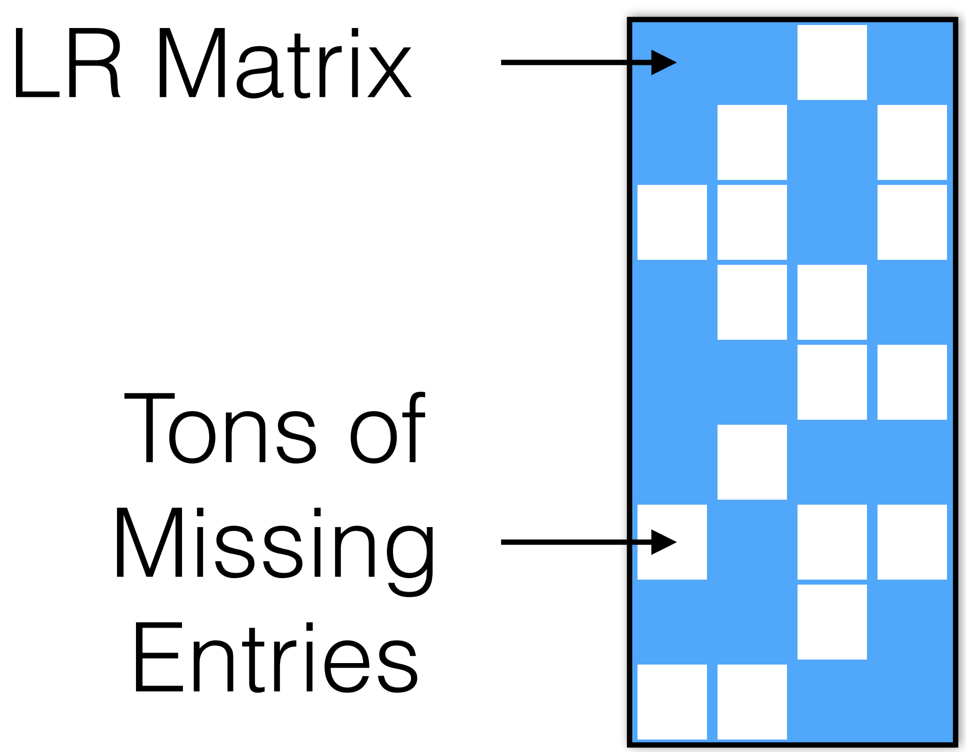

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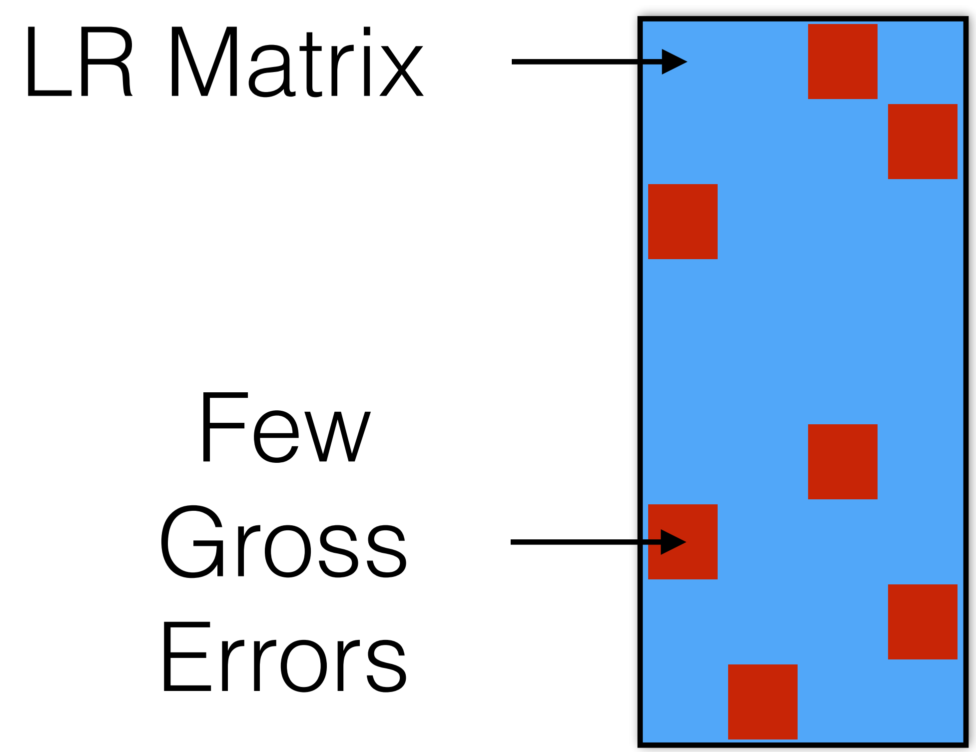

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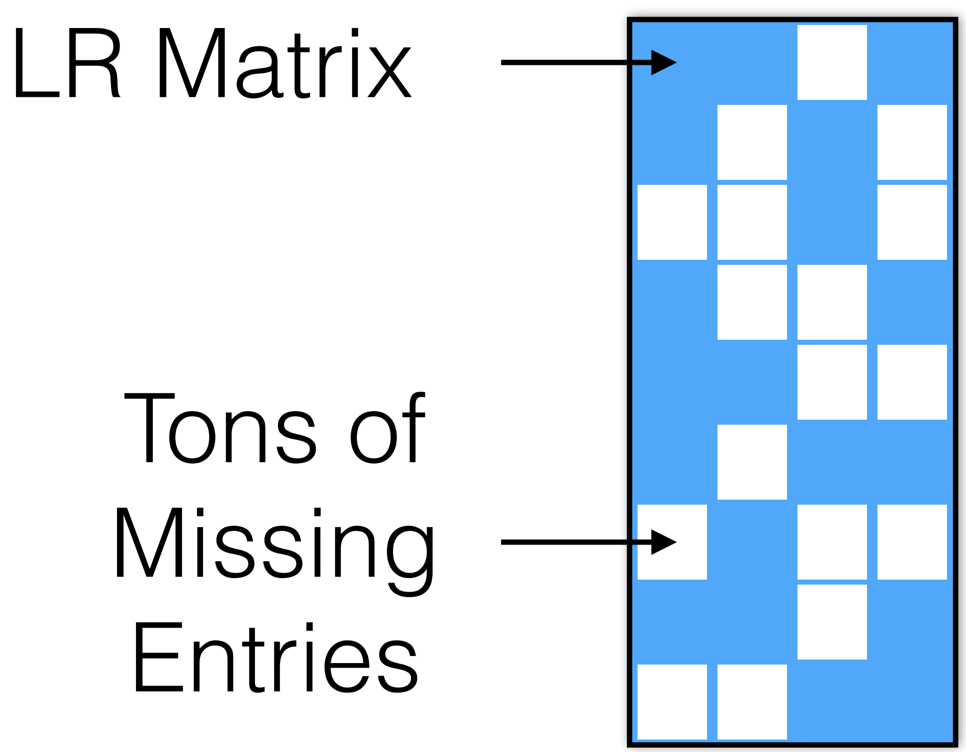
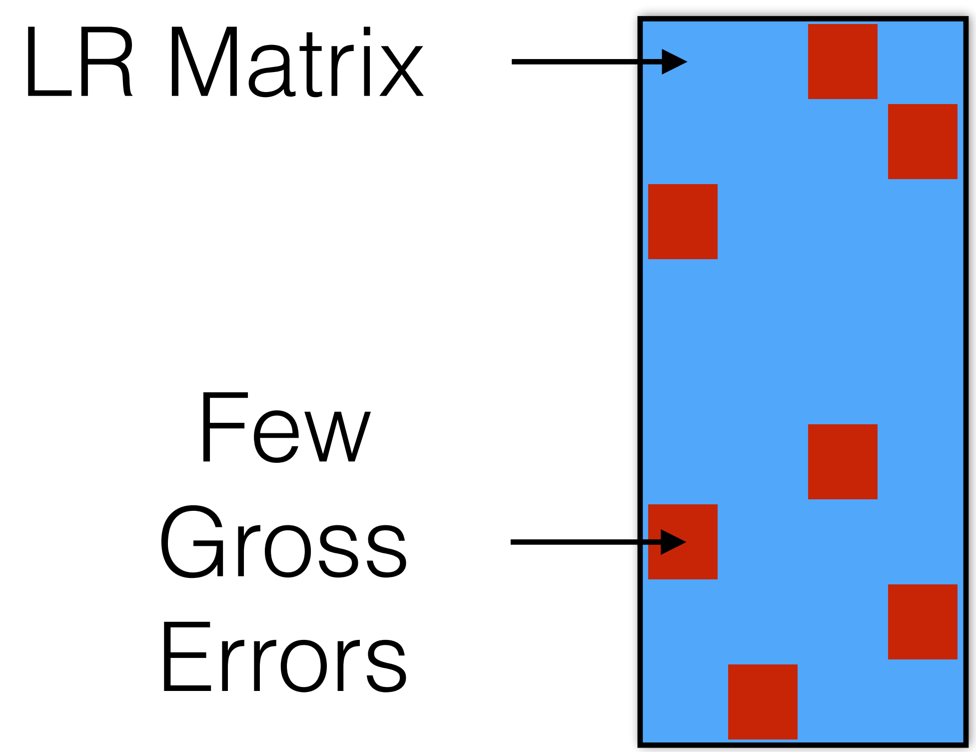


- ☑ Know Locations
- ☐ Don't know values

RPCA

(Robust Principal Component Analysis)

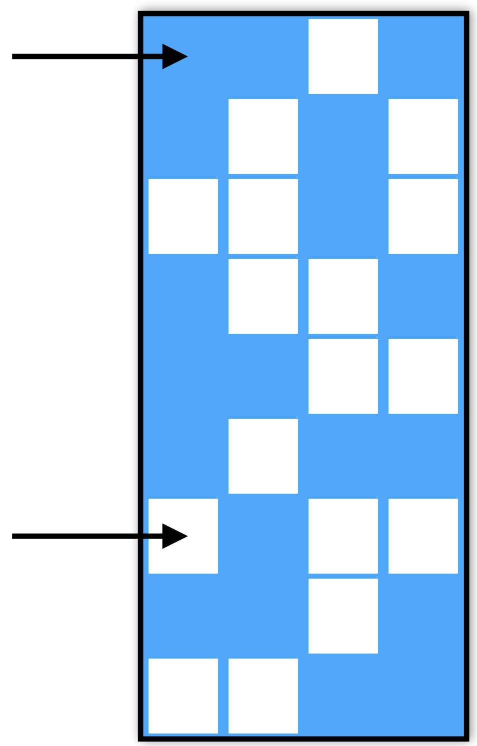


LRMC

(Low-Rank Matrix Completion)

LR Matrix

Tons of Missing Entries



☑ Know Locations

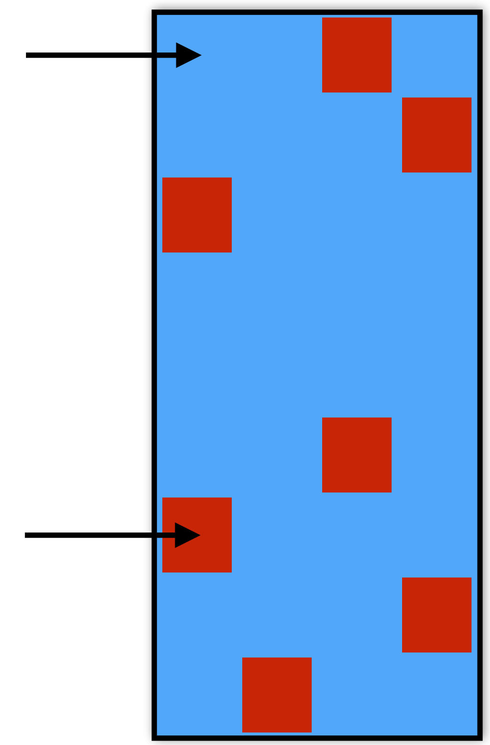
☐ Don't know values

RPCA

(Robust Principal Component Analysis)

LR Matrix

Few Gross Errors



☐ Don't know Locations

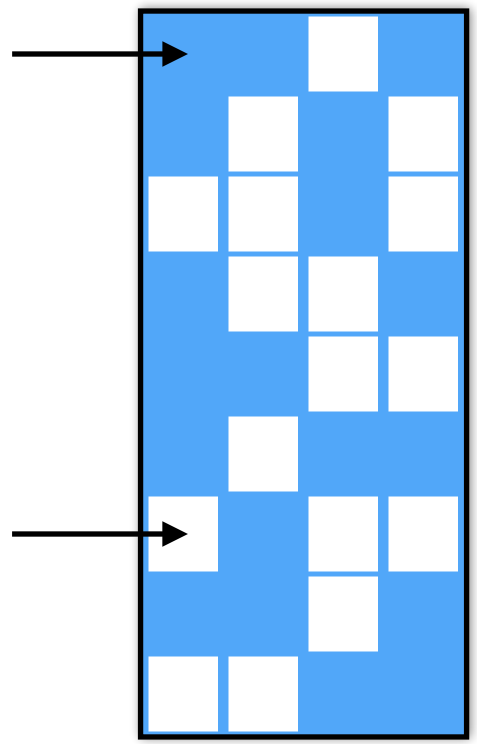
☑ Know all values

LRMC

(Low-Rank Matrix Completion)

LR Matrix

Tons of Missing Entries



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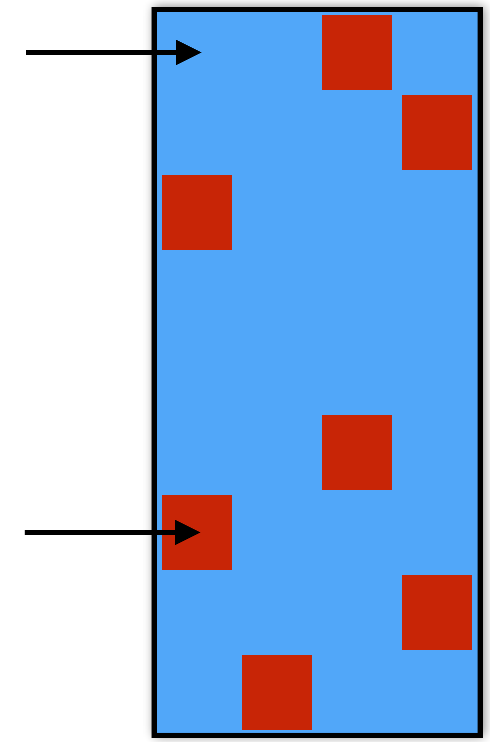
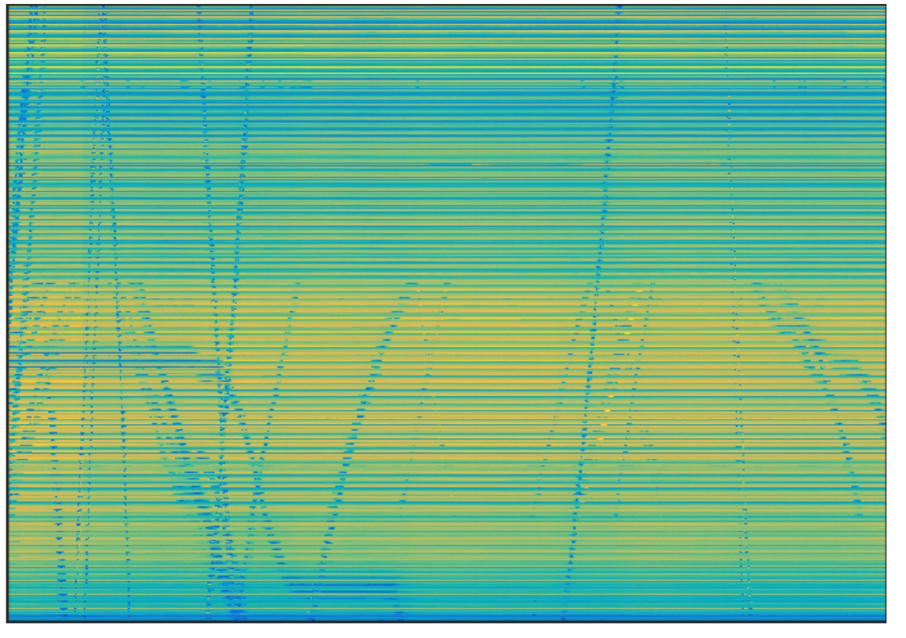


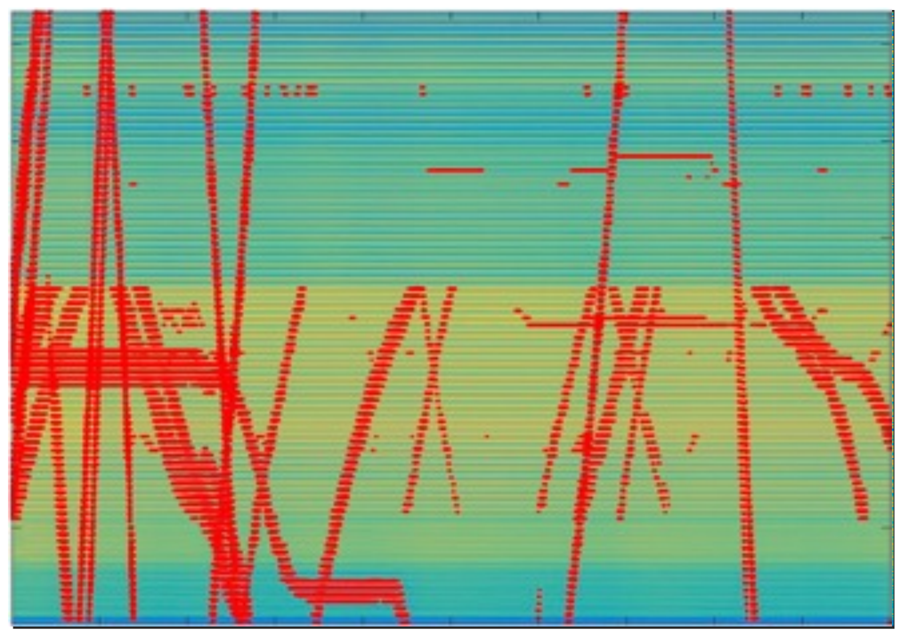
- ☐ Don't know Locations
- ☑ Know all values

Common goal: find the subspace

Background segmentation



Background segmentation



Background segmentation

Existing Approaches

$$\text{minimize} \quad \|\mathbf{L}\|_* + \lambda\|\mathbf{S}\|_1$$
$$\text{subject to} \quad \mathbf{X} = \mathbf{L} + \mathbf{S}$$

[1] F. De La Torre and M. Black, *A framework for robust subspace learning*, International Journal of Computer Vision, 2003.

[2] Q. Ke and T. Kanade, *Robust L_1 norm factorization in the presence of outliers and missing data by alternative convex programming*, IEEE Conference on Computer Vision and Pattern Recognition, 2005.

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[5] E. Candès, X. Li , Y. Ma and J. Wright, *Robust principal component analysis?*, Journal of the ACM, 2011.

[6] V. Chandrasekaran, S. Sanghavi, P. Parrilo and A. Willsky, *Rank-sparsity incoherence for matrix decomposition*, SIAM Journal on Optimization, 2011.

[7] L. Mackey, A. Talwalkar and M. Jordan, *Divide-and-conquer matrix factorization*, Advances in Neural Information Processing Systems, 2011.

[8] M. Rahmani and G. Atia, *A subspace learning approach for high dimensional matrix decomposition with efficient column/row sampling*, International Conference on Machine Learning, 2016.

[9] T. Bouwmans and E. Zahzah, *Robust PCA via principal component pursuit: a review for a comparative evaluation in video surveillance*, Computer Vision and Image Understanding, 2014.

[10] Z. Lin, M. Chen, L. Wu, and Y. Ma, *The augmented Lagrange multiplier method for exact recovery of corrupted low-rank matrices*, University of Illinois at Urbana-Champaign Technical Report, 2009.

[11] Z. Lin, R. Liu and Z. Su, *Linearized alternating direction method with adaptive penalty for low rank representation*, Advances in Neural Information Processing Systems, 2011.

[12] X. Yuan and J. Yang, *Sparse and low-rank matrix decomposition via alternating direction methods*, 2009.

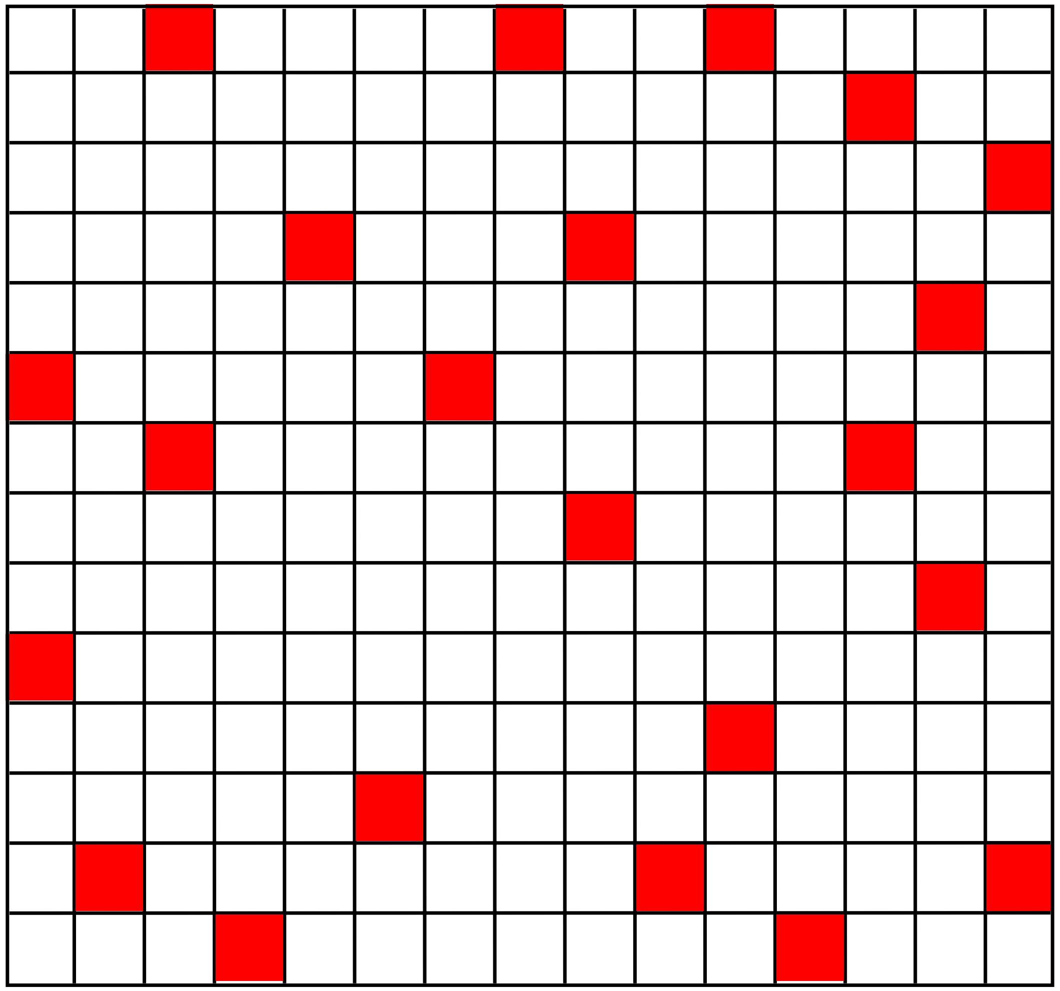
[13] Z. Lin, A. Ganesh, J. Wright, L. Wu, M. Chen and Y. Ma, *Fast convex optimization algorithms for exact recovery of a corrupted low-rank matrix*, Computational Advances in Multi-Sensor Adaptive Processing, 2009.

[14] Y. Shen, Z. Wen, and Y. Zhang. Augmented Lagrangian Alternating Direction Method for Matrix Separation based on Low-Rank Factorization, Optimization Methods and Software, 2011.

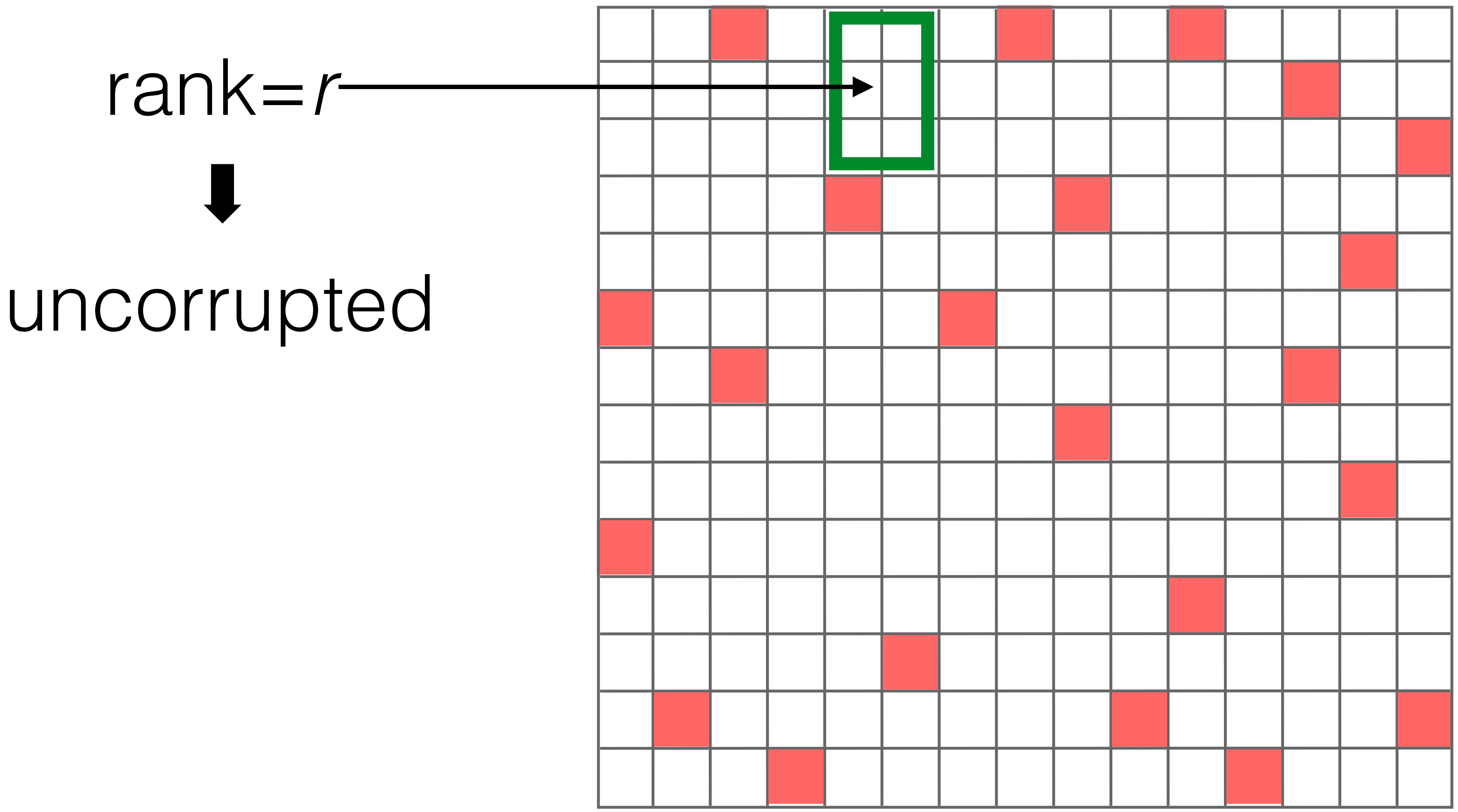
Using our Theory

Totally different way to think about the problem

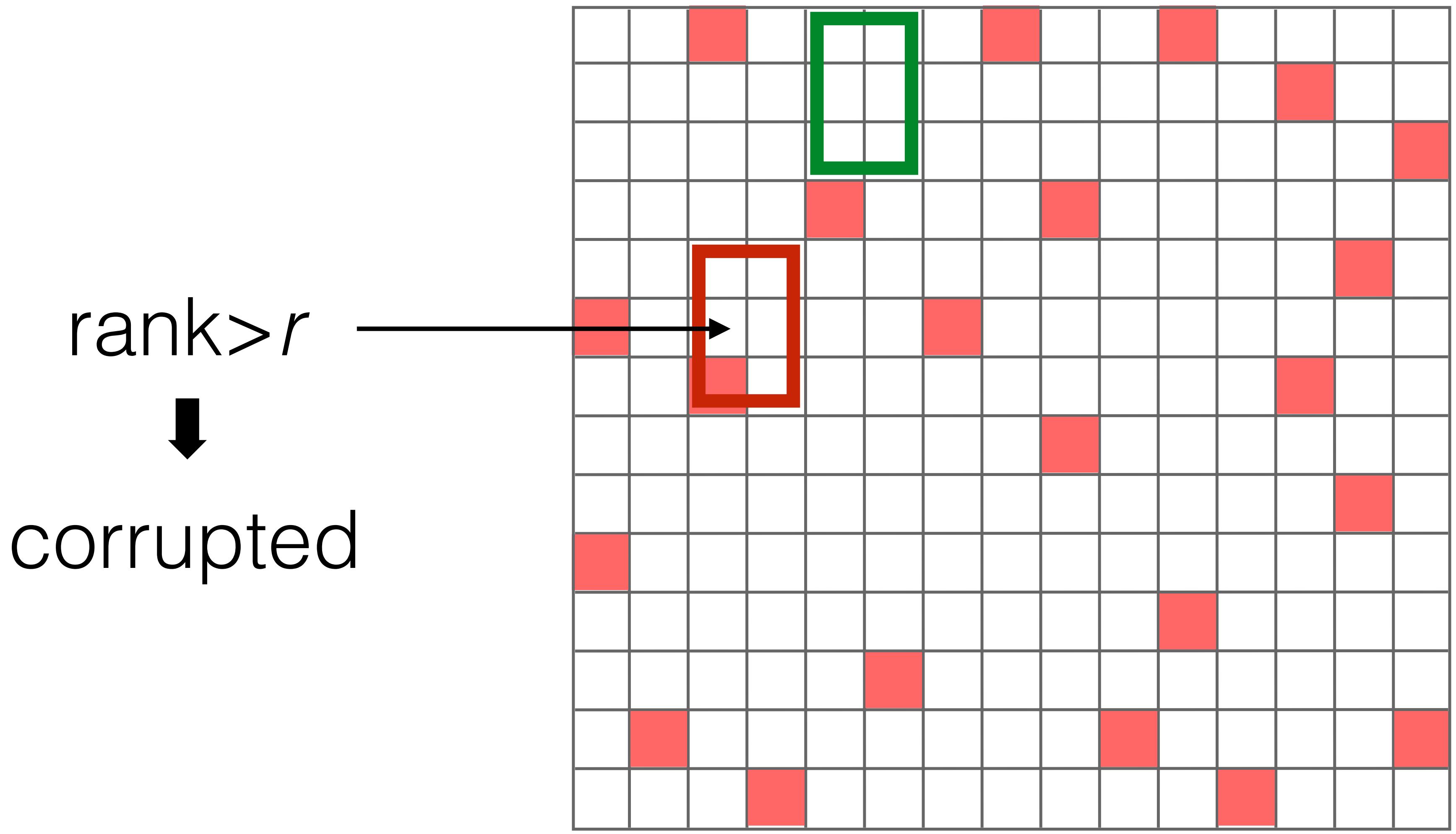
Use incomplete-data tricks



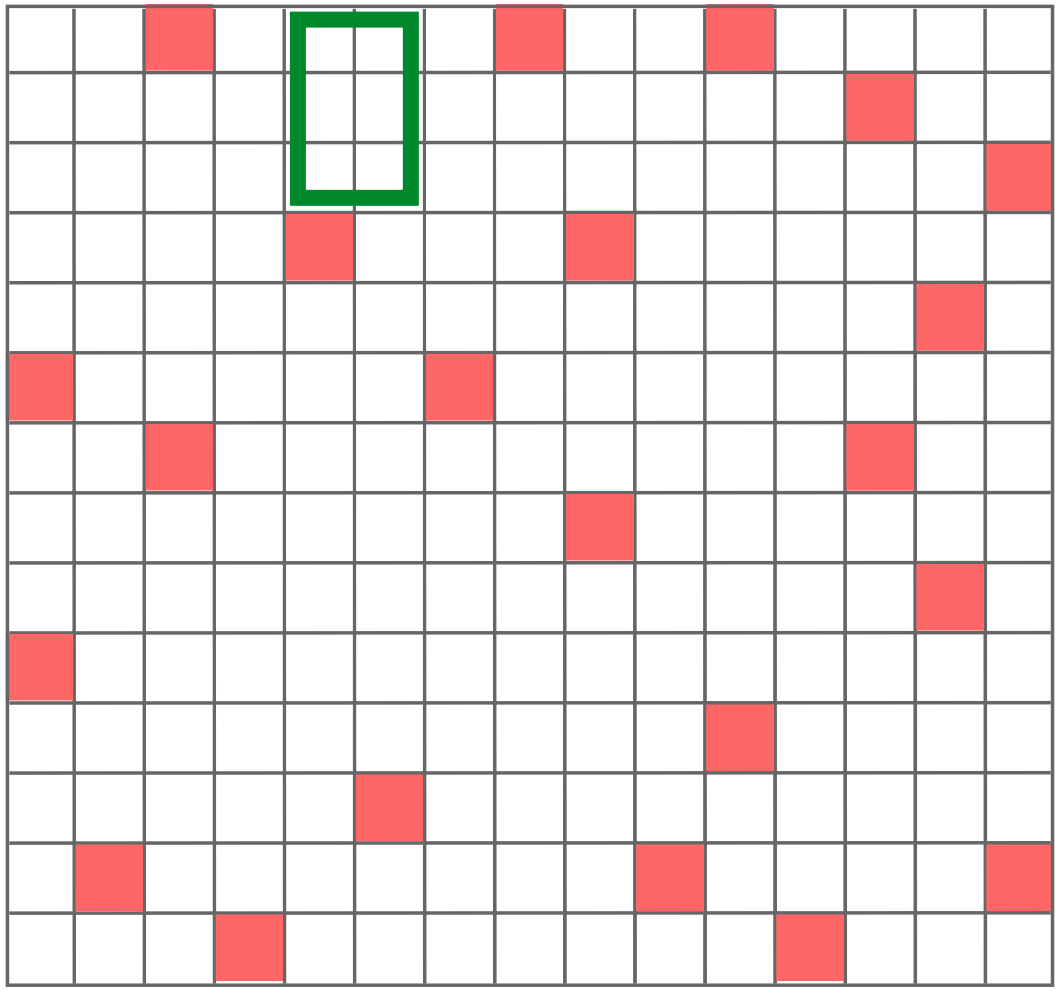
Use incomplete-data tricks



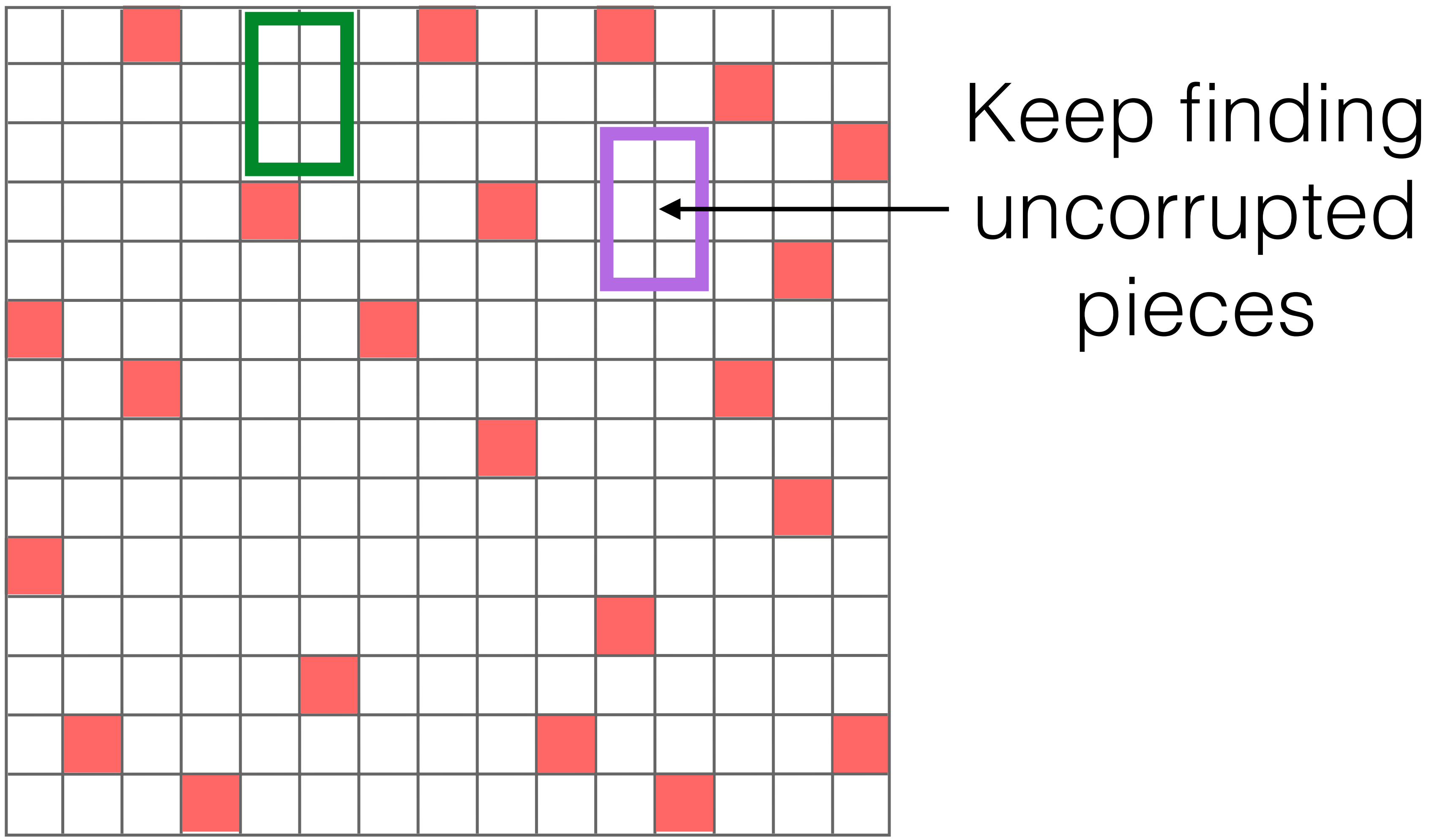
Use incomplete-data tricks



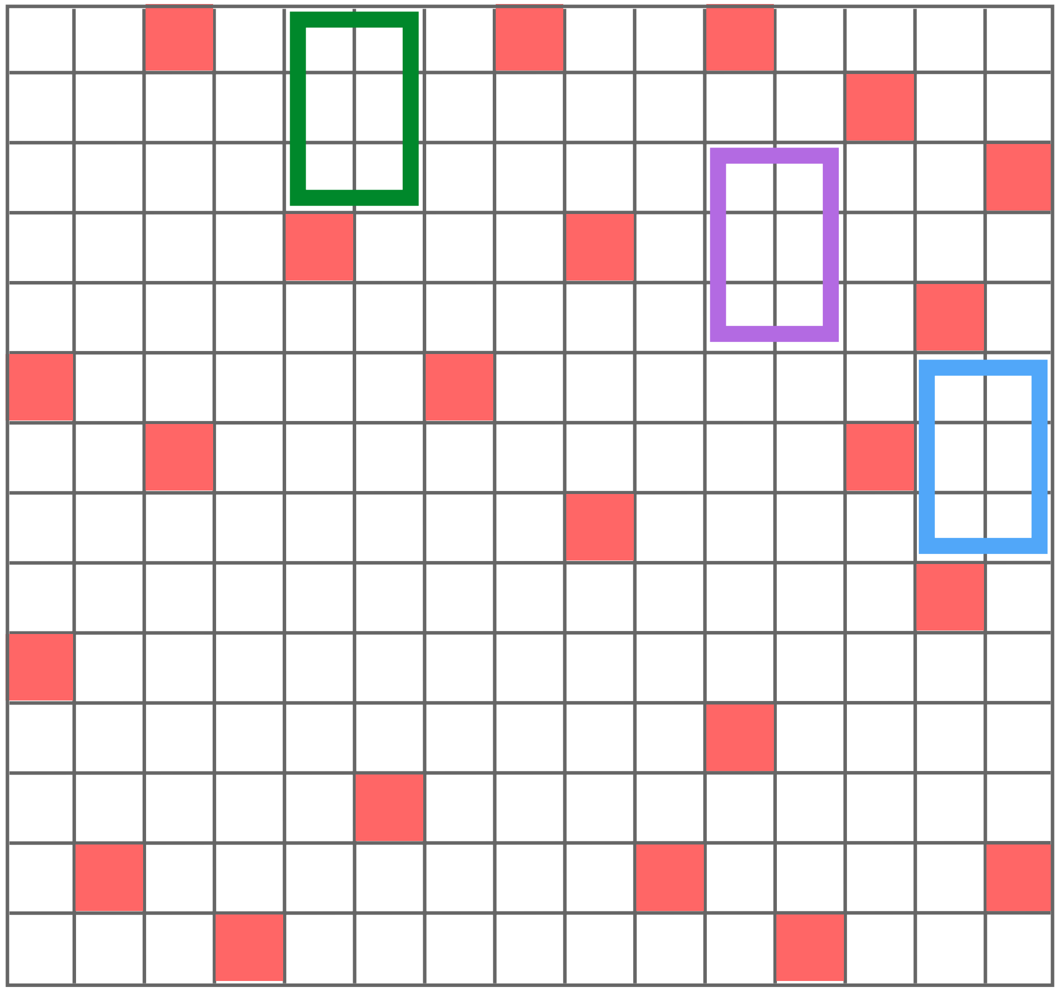
Use incomplete-data tricks



Use incomplete-data tricks

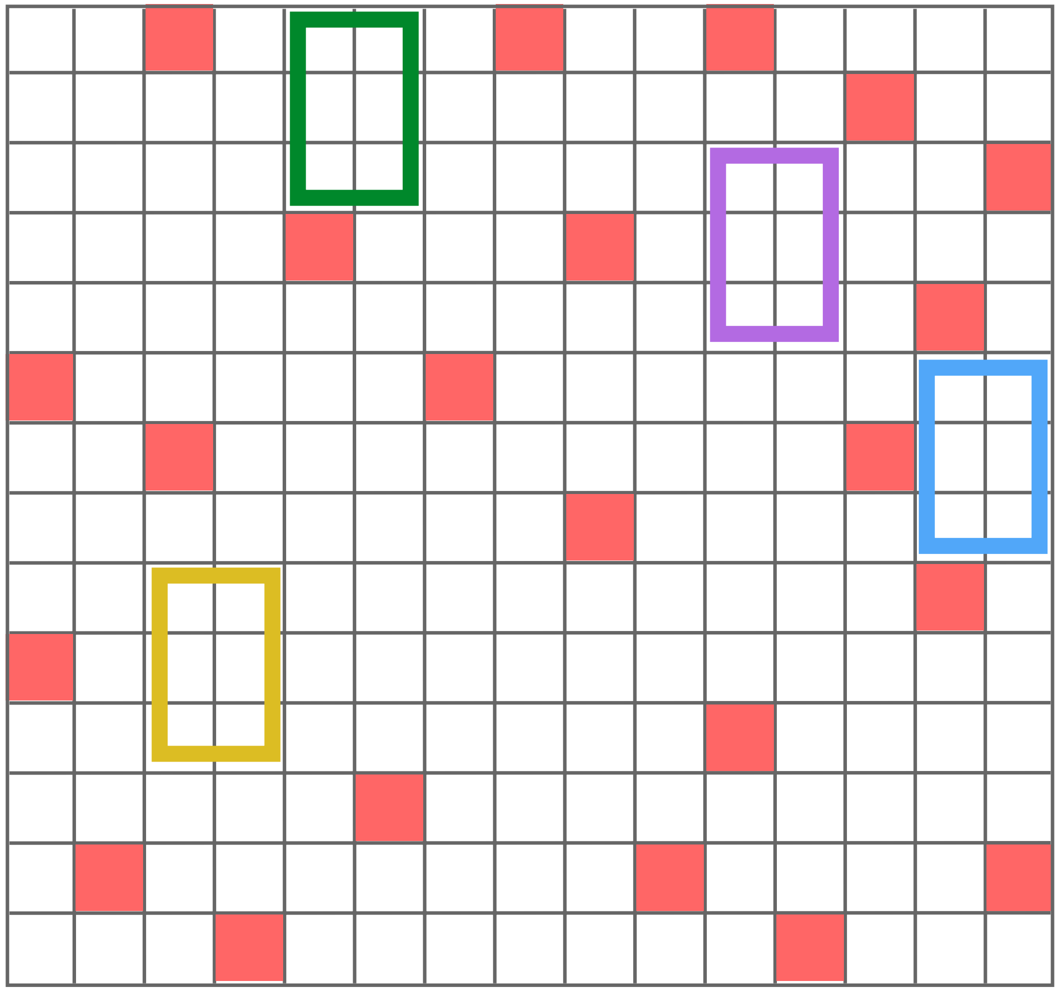


Use incomplete-data tricks



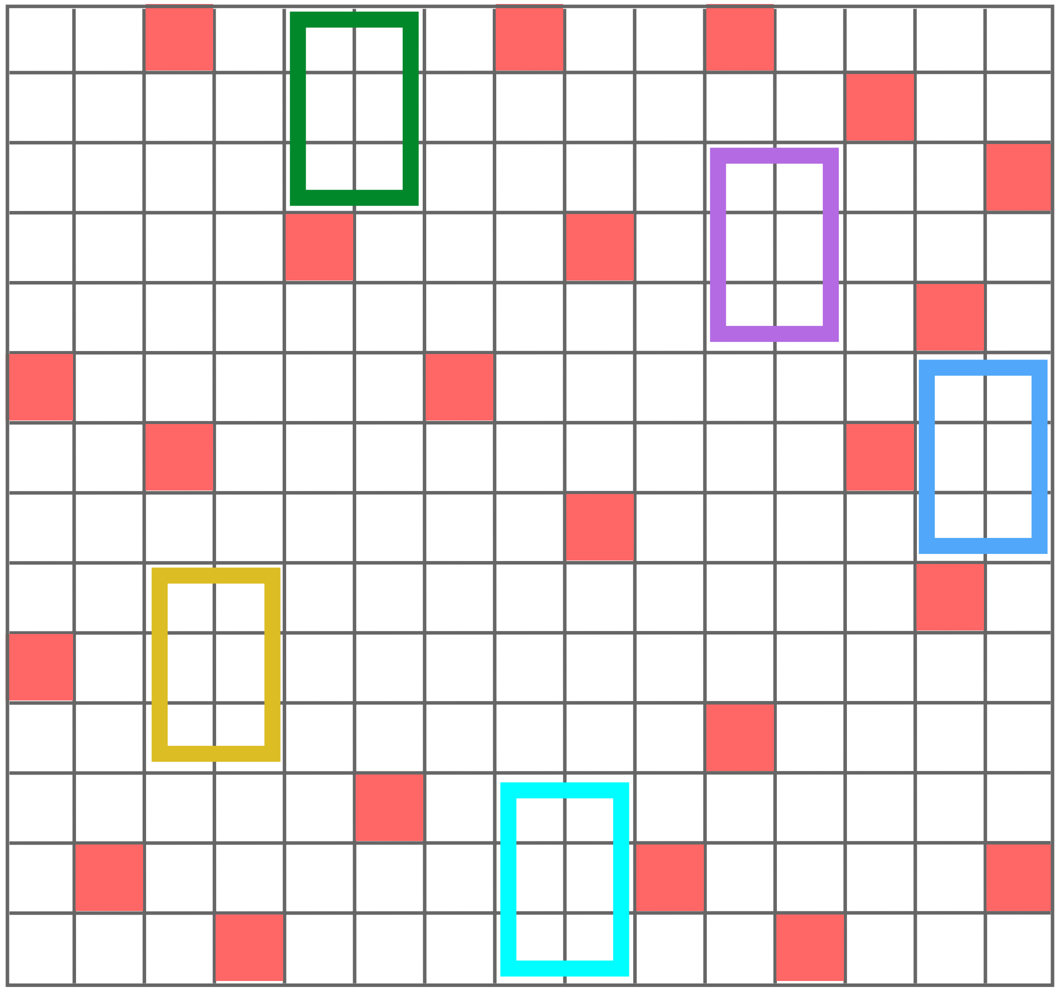
Keep finding uncorrupted pieces

Use incomplete-data tricks



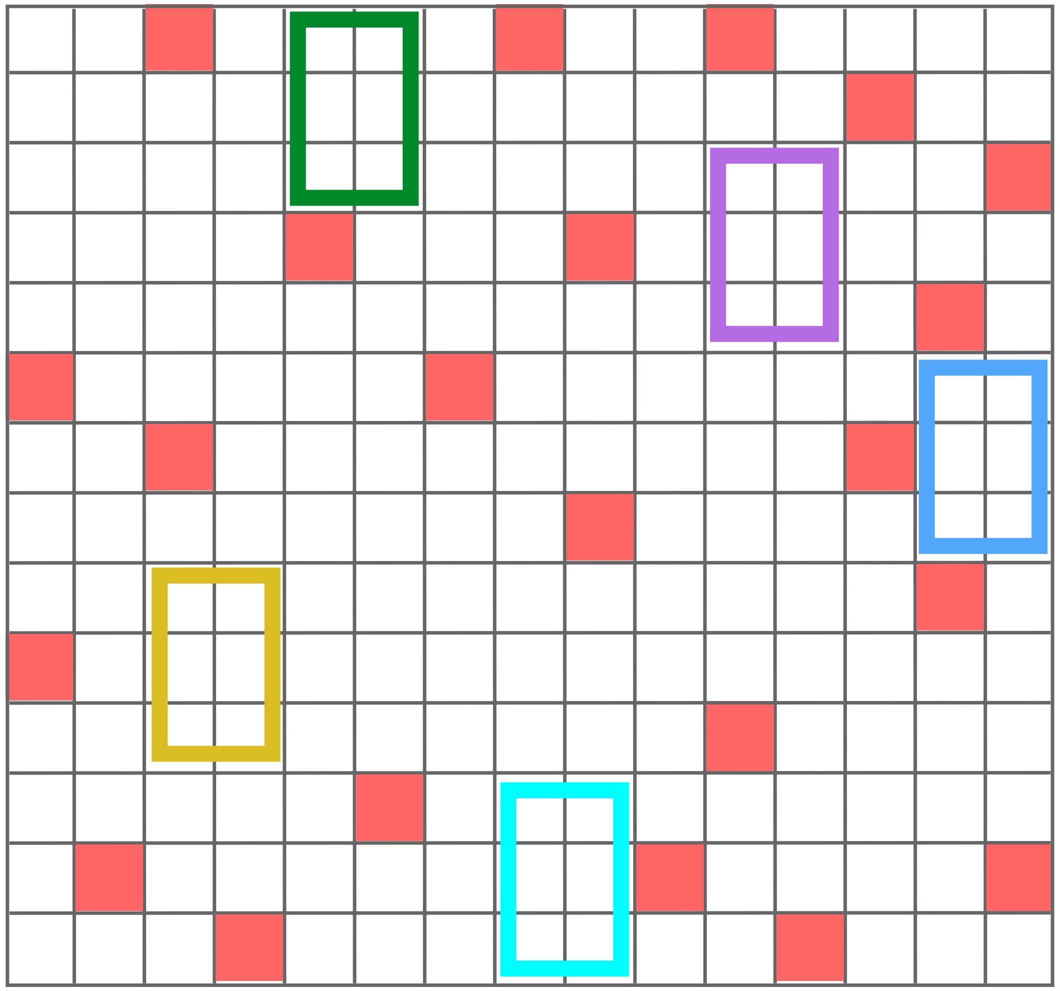
Keep finding uncorrupted pieces

Use incomplete-data tricks



Keep finding uncorrupted pieces

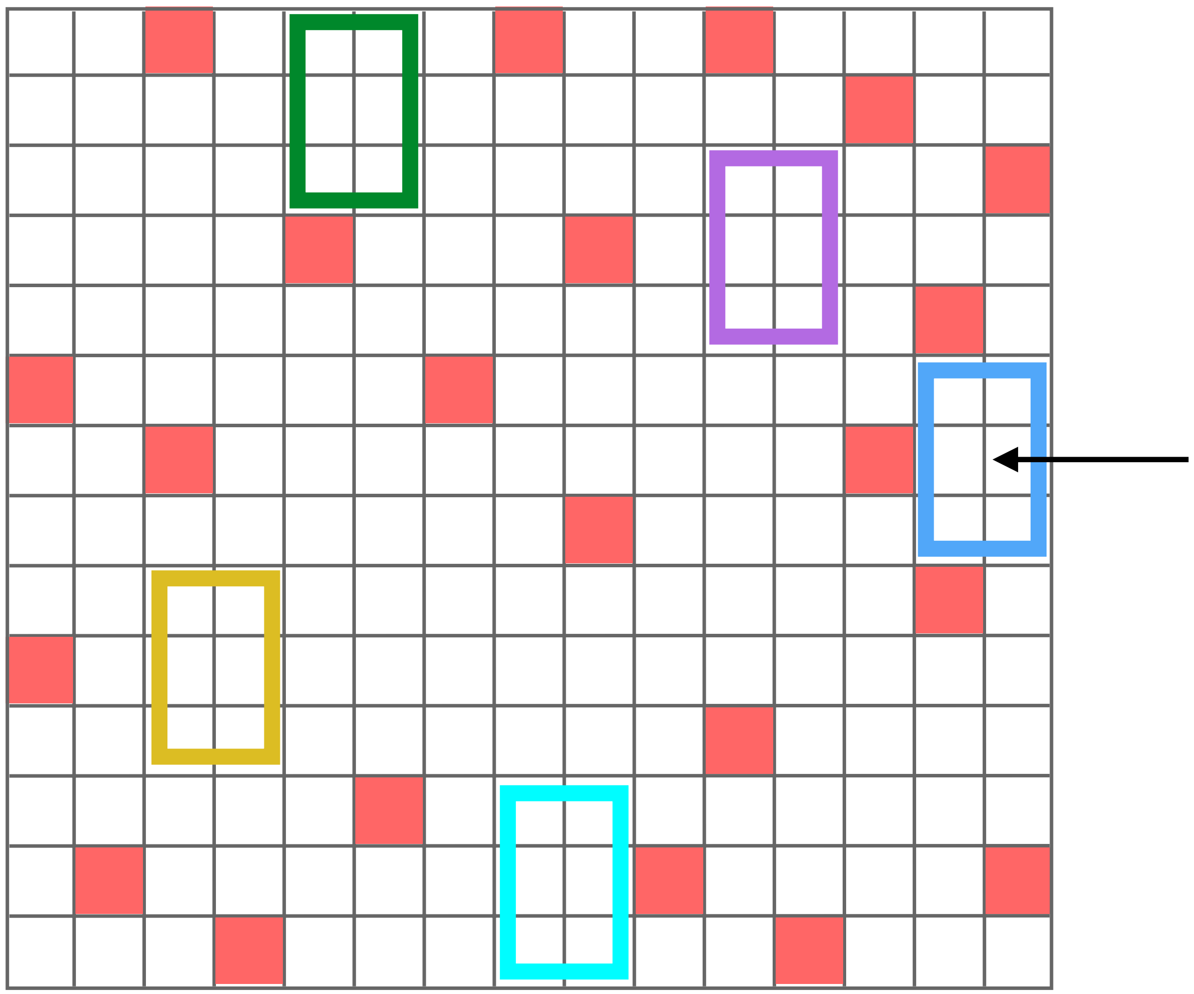
Use incomplete-data tricks



Keep finding uncorrupted pieces

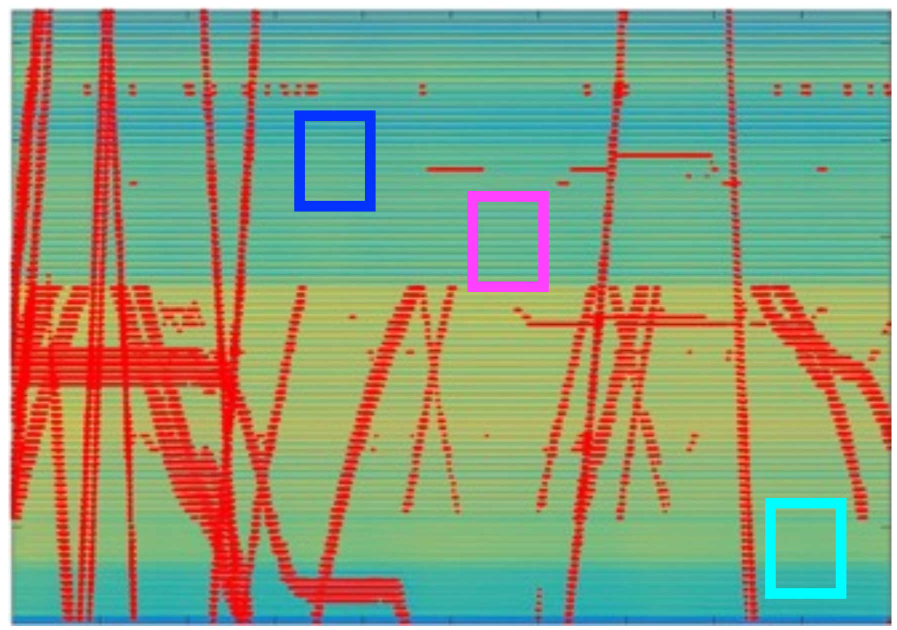
If pieces are *observed in the right places*, we can find the subspace

Use incomplete-data tricks



It gets better:
For these sorts of pieces, polynomials become linear.

If pieces are *observed in the right places*,
we can find the subspace efficiently



Background segmentation

Original Frame	Our Work	RPCA-ALM
	[3] Pimentel et. al (2017)	RPCA-ALM (Lin et. al, 2011-2016)

In many cases, similar results

Original Frame

Our Work

[3] Pimentel et. al (2017)

RPCA-ALM

RPCA-ALM (Lin et. al, 2011-2016)

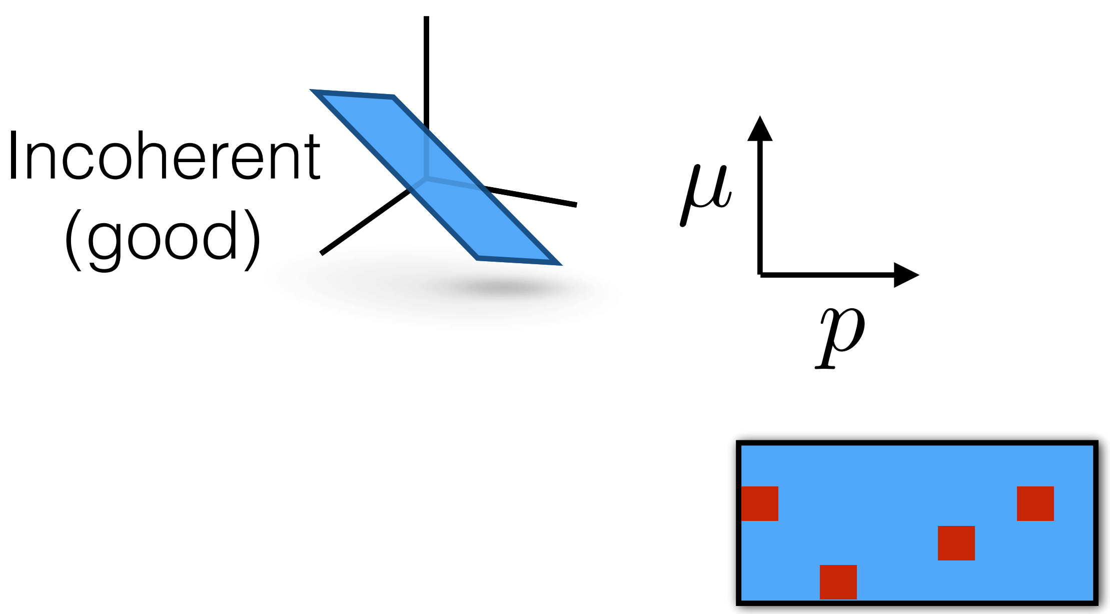
In other cases, better

Original Frame

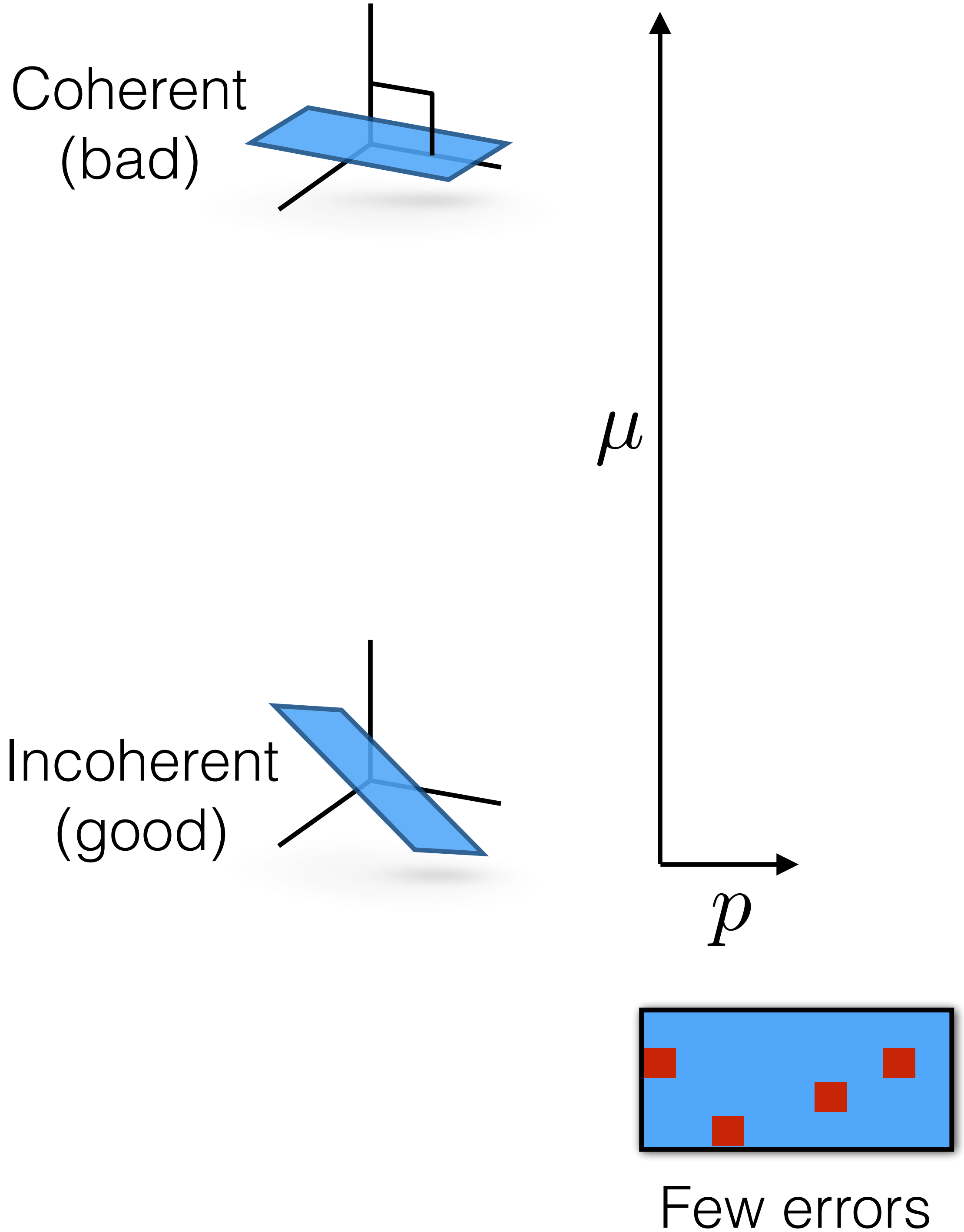
Our Work
[3] Pimentel et. al (2017)

RPCA-ALM
RPCA-ALM (Lin et. al, 2011-2016)

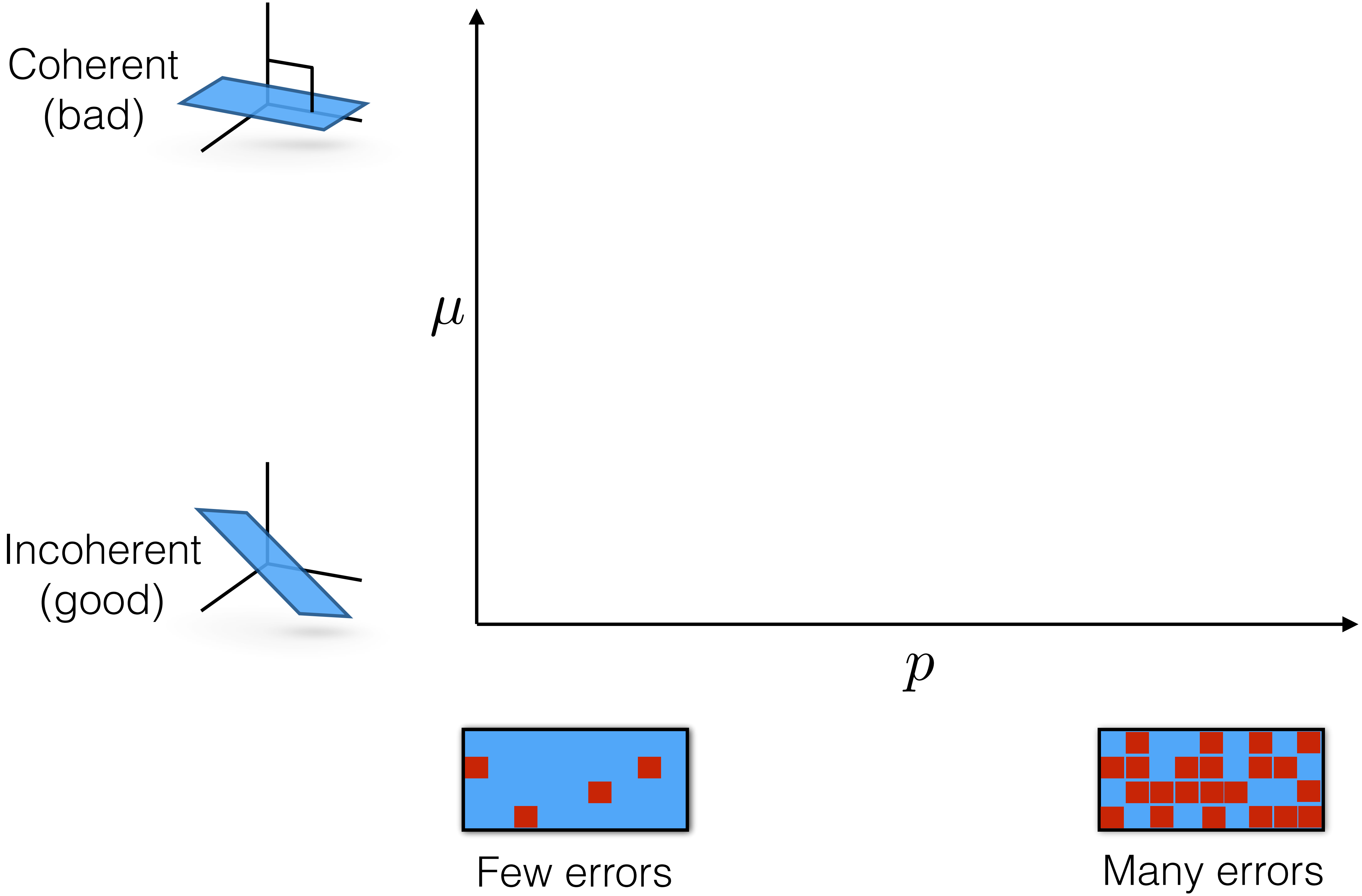
In other cases, better



Performance Analysis

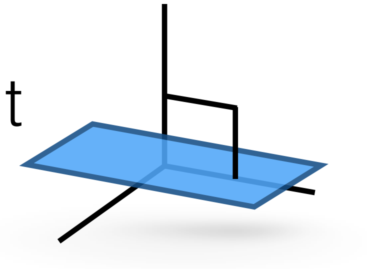


Performance Analysis

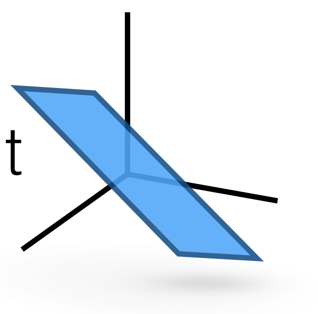


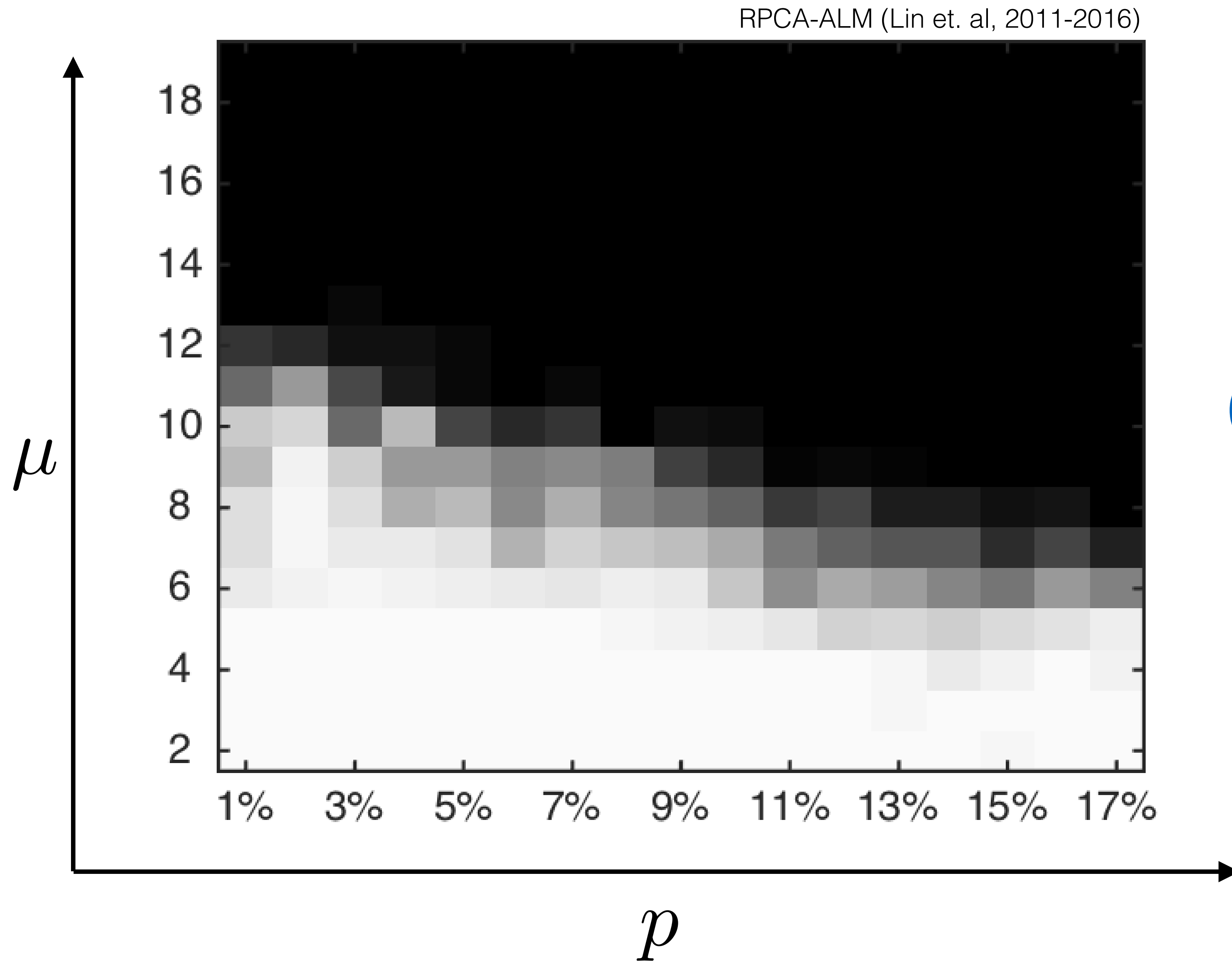
Performance Analysis

Coherent
(bad)

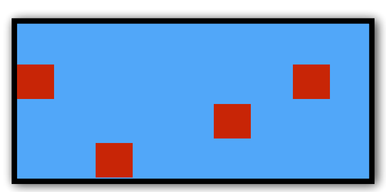


Incoherent
(good)

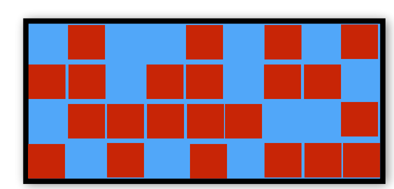




(the lighter
the better)



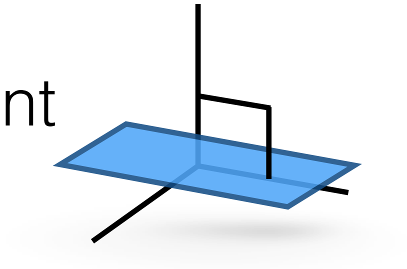
Few errors

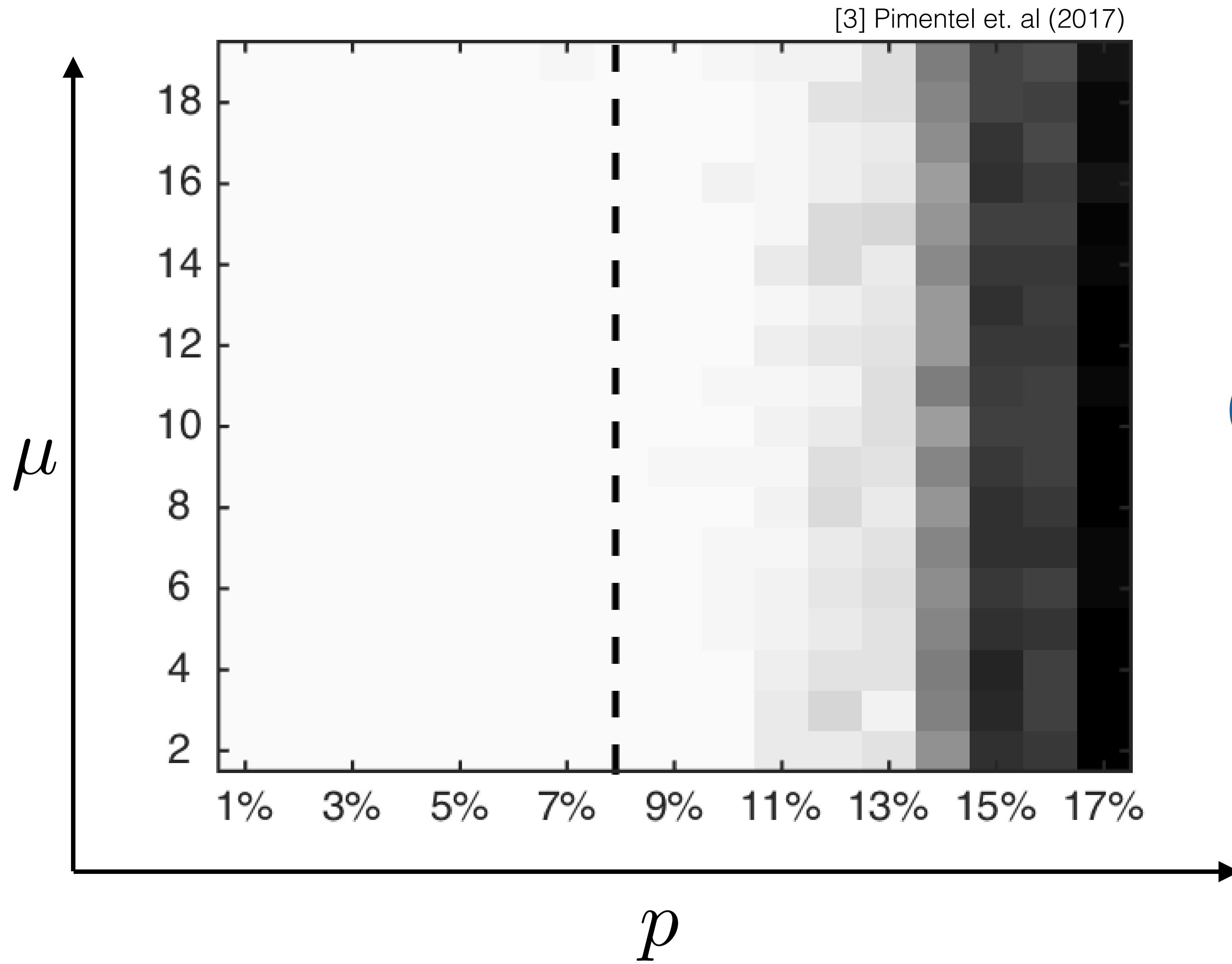


Many errors

Performance Analysis

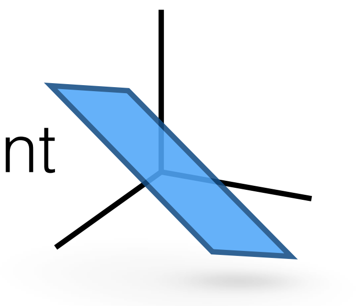
Coherent
(bad)

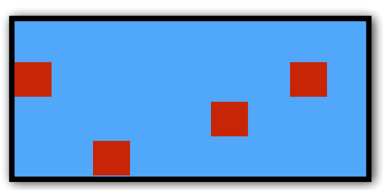




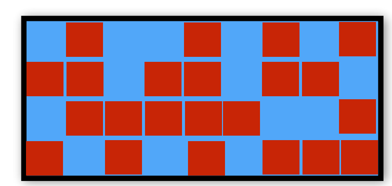
(the lighter the better)

Incoherent
(good)



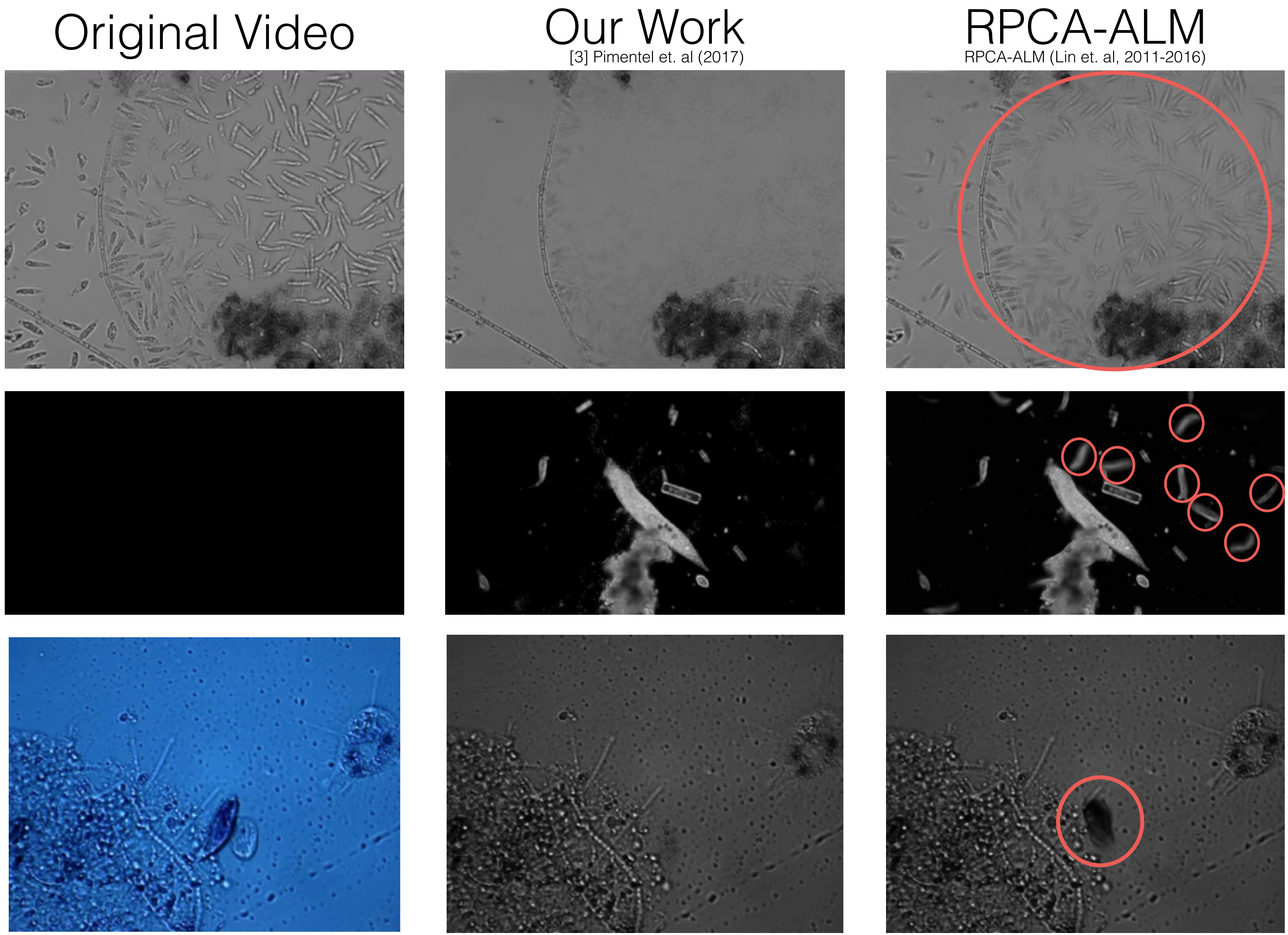


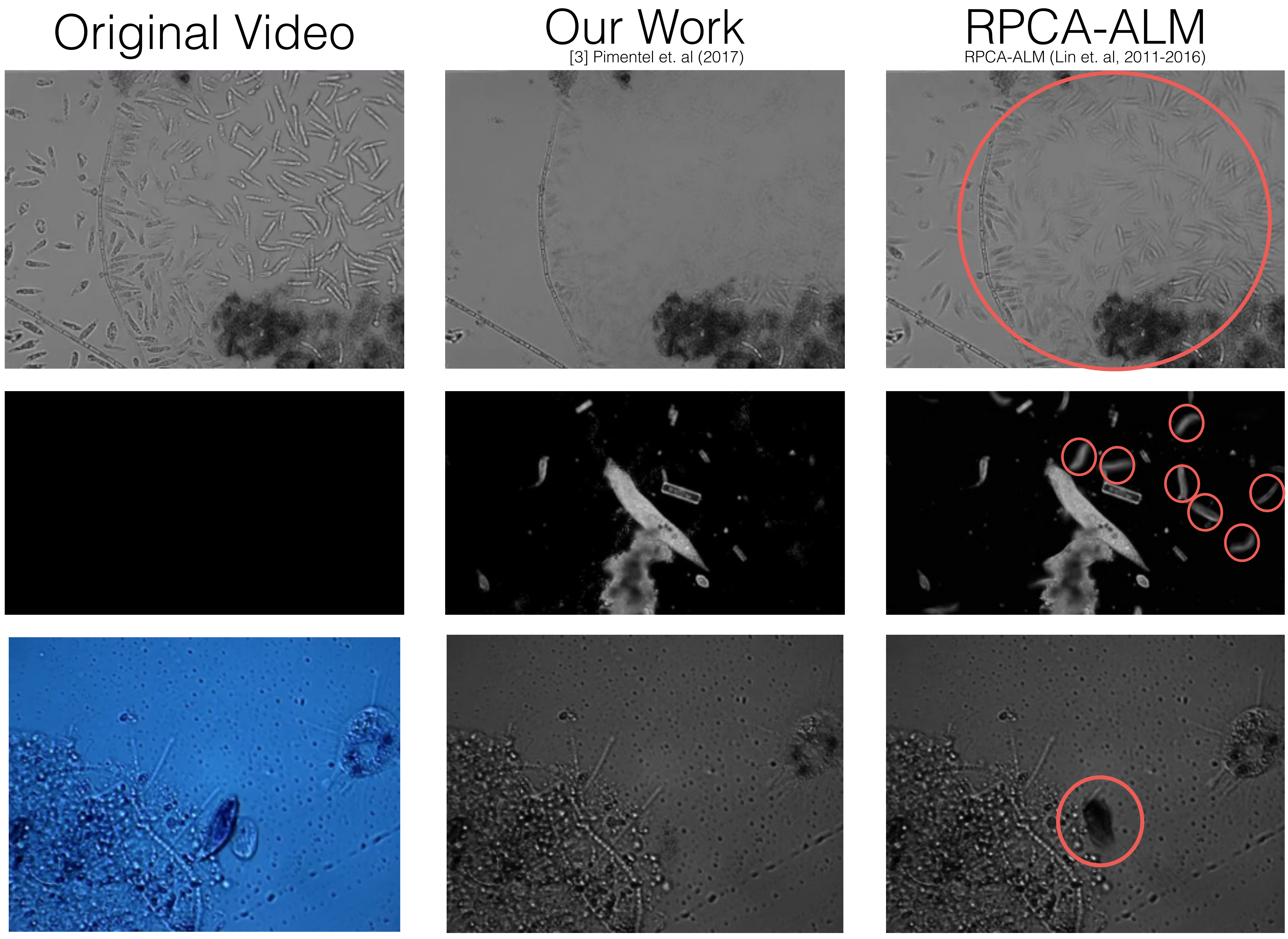
Few errors



Many errors

Performance Analysis

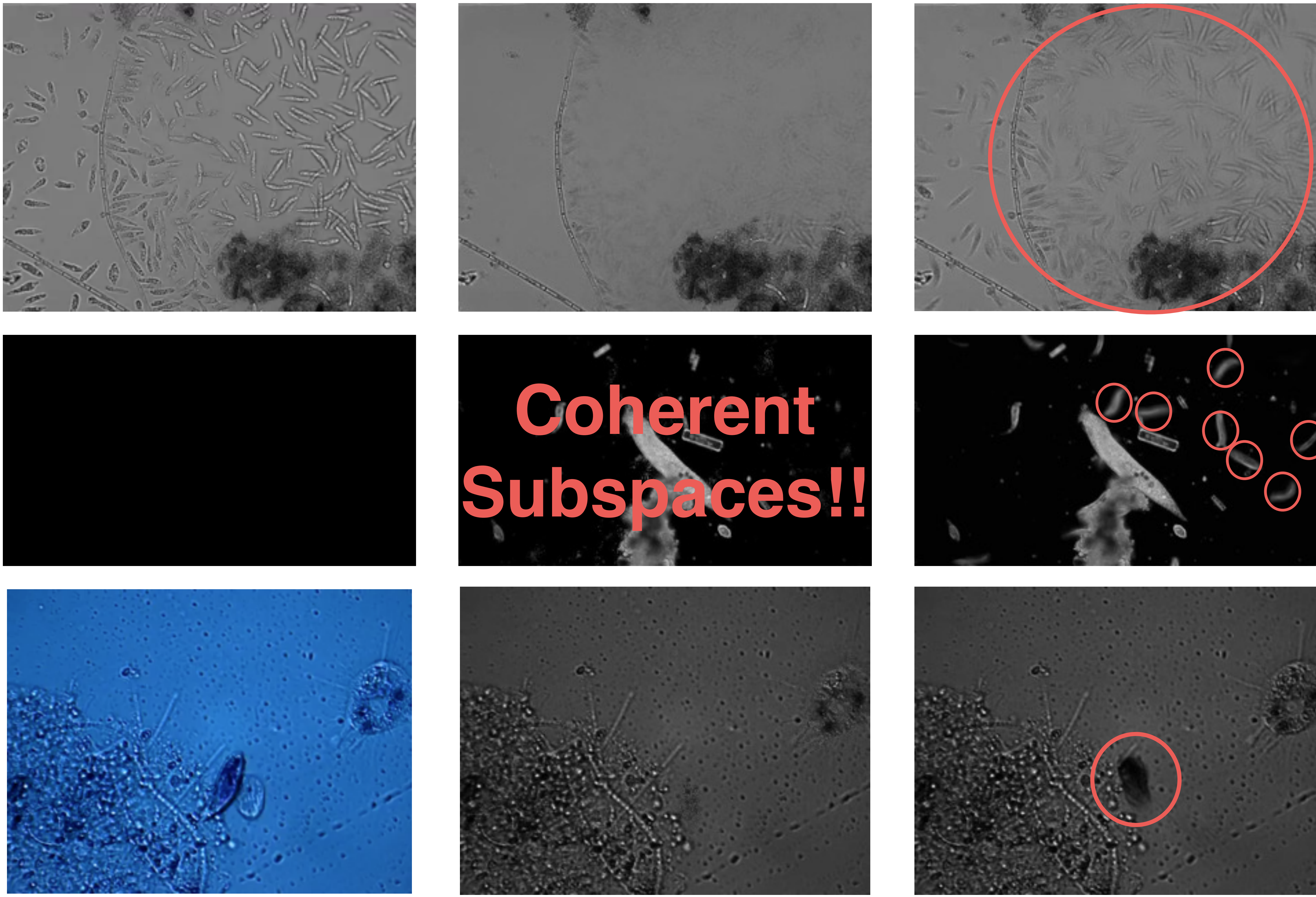

Original Video
Our Work
[3] Pimentel et. al (2017)
RPCA-ALM
RPCA-ALM (Lin et. al, 2011-2016)

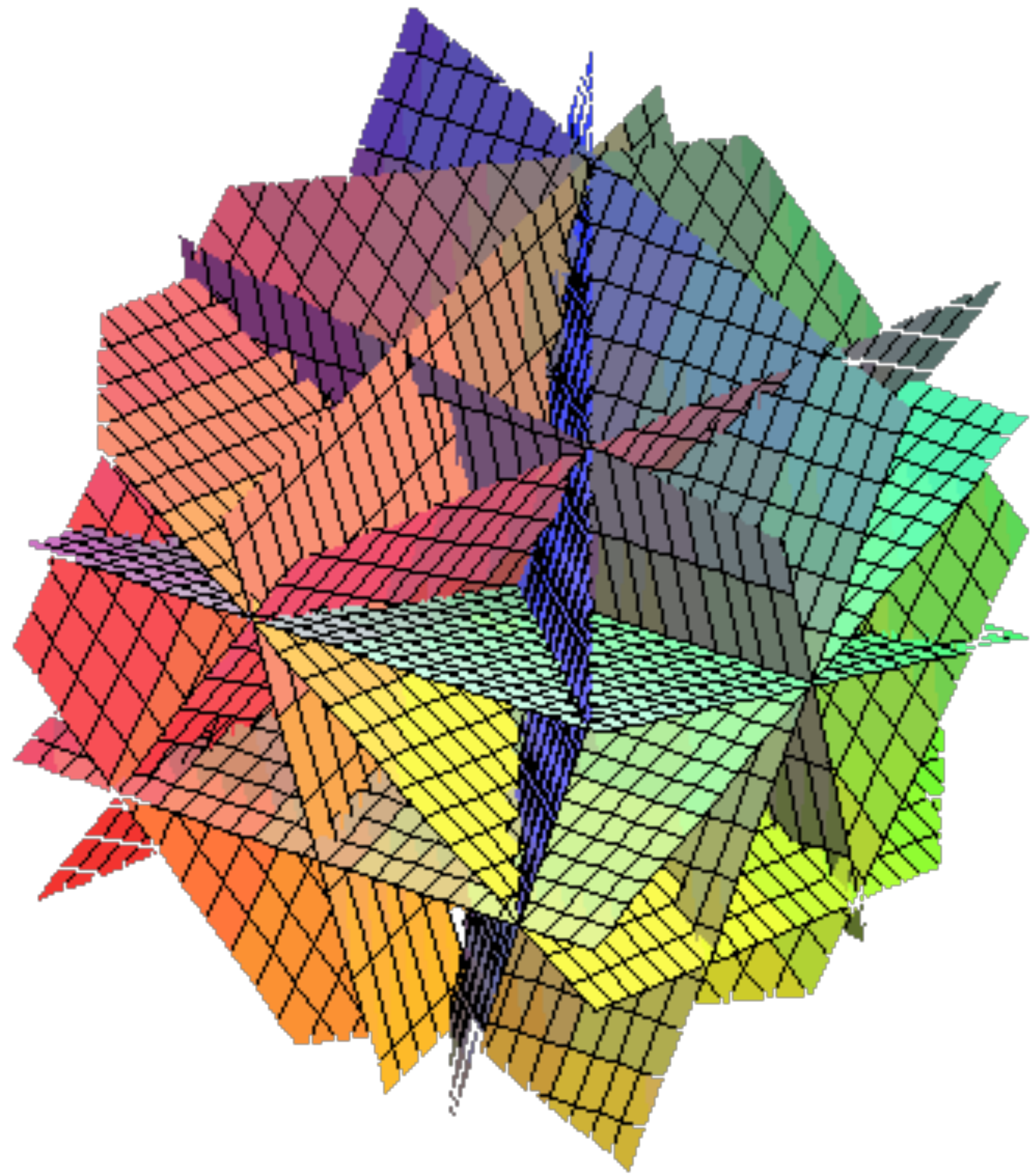


Original Video
Our Work
[3] Pimentel et. al (2017)
RPCA-ALM
RPCA-ALM (Lin et. al, 2011-2016)

Original Video | Our Work | RPCA-ALM

[3] Pimentel et. al (2017) | RPCA-ALM (Lin et. al, 2011-2016)





Beyond one Subspace

Multiple subspaces

We want to find them all!

Multiple subspaces

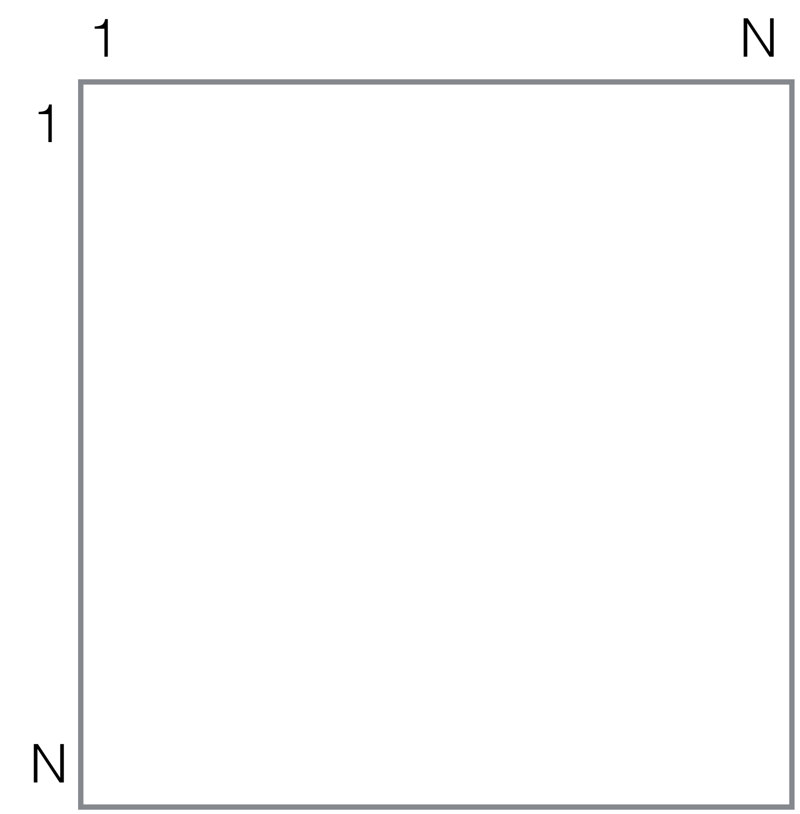
We want to find them all!

Multiple subspaces

We want to find them all!

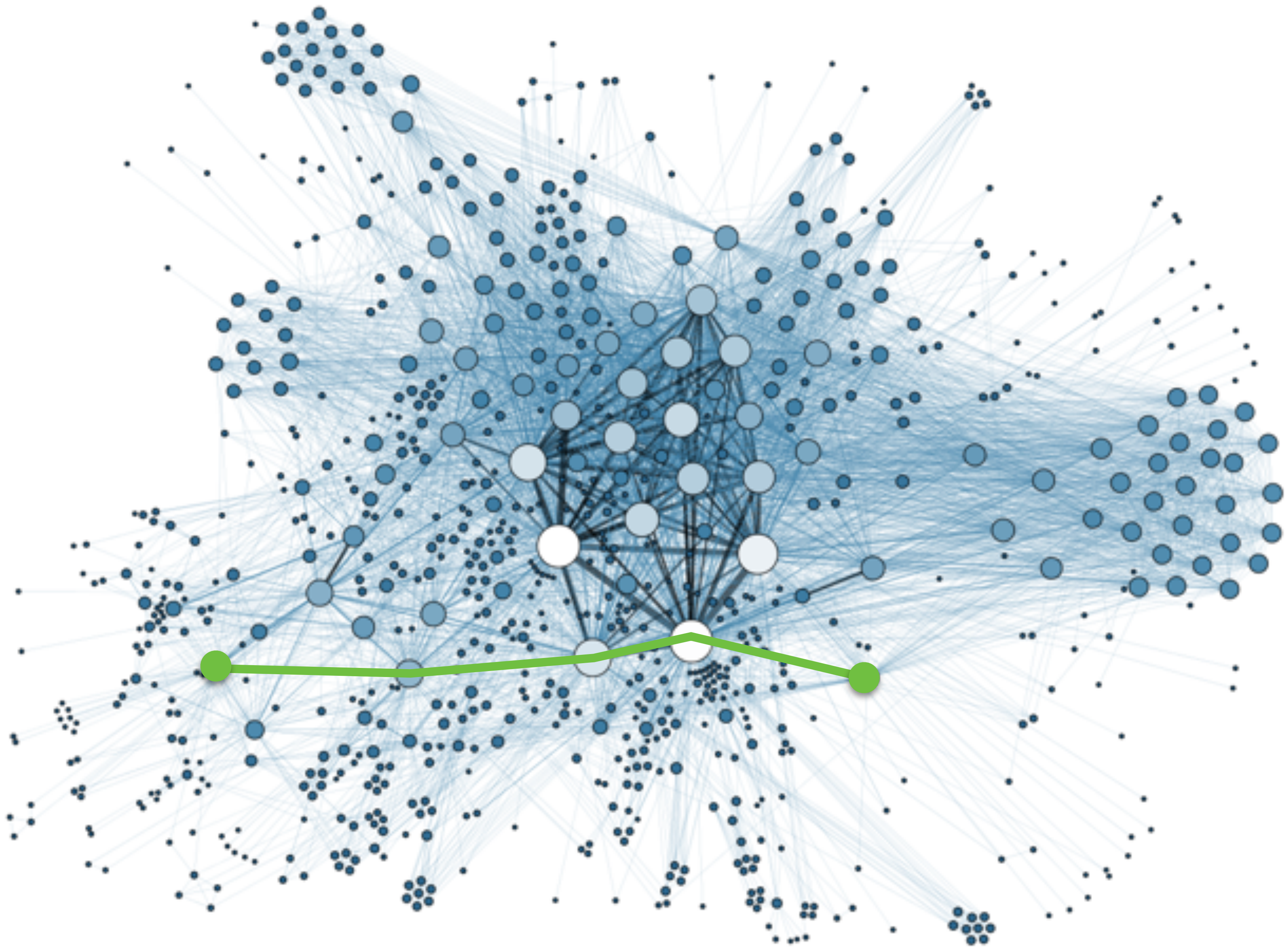
Multiple subspaces

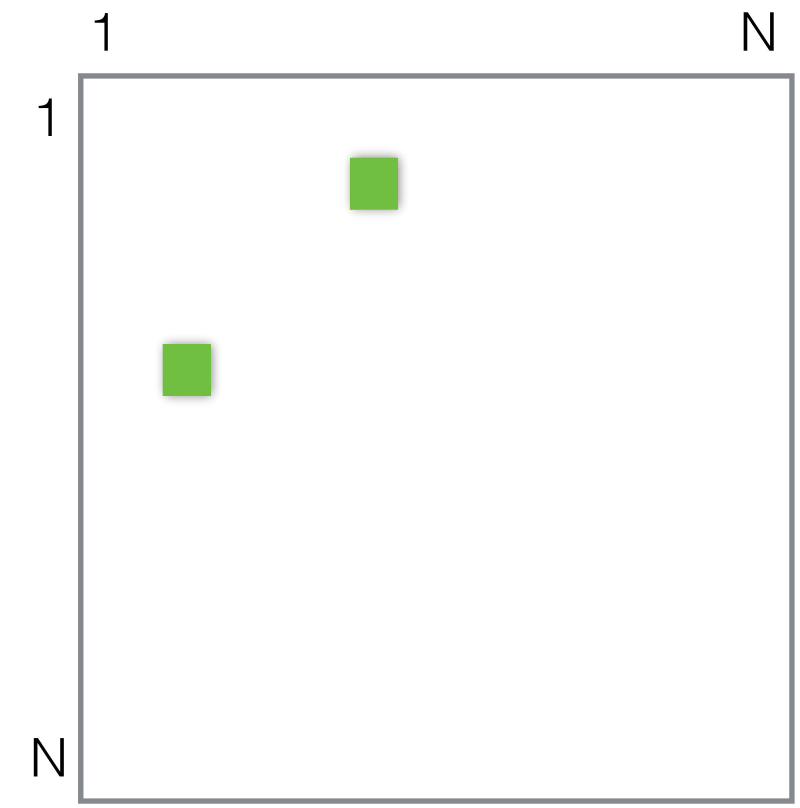
We want to find them all!



Multiple subspaces

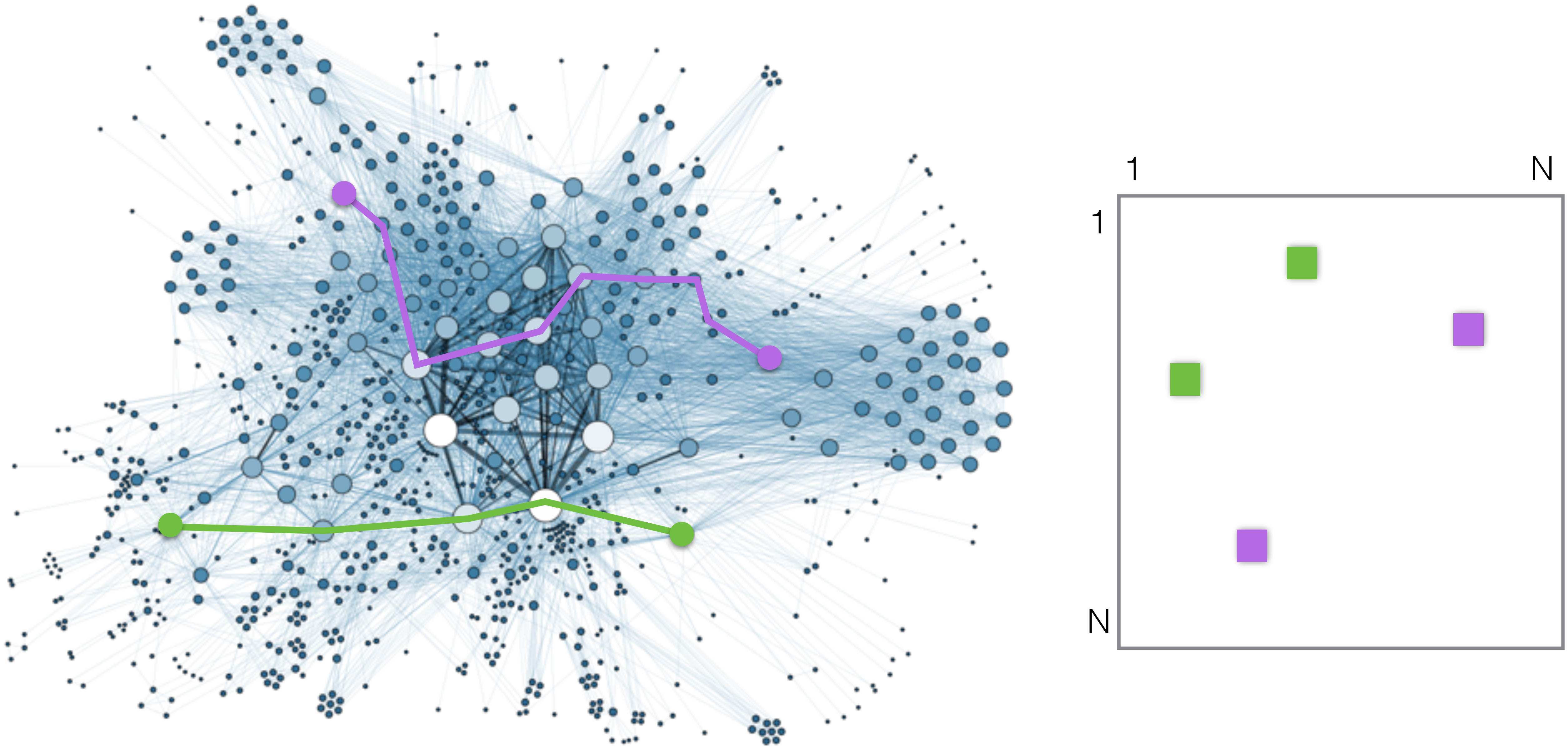
We want to find them all!





Multiple subspaces

We want to find them all!



Multiple subspaces

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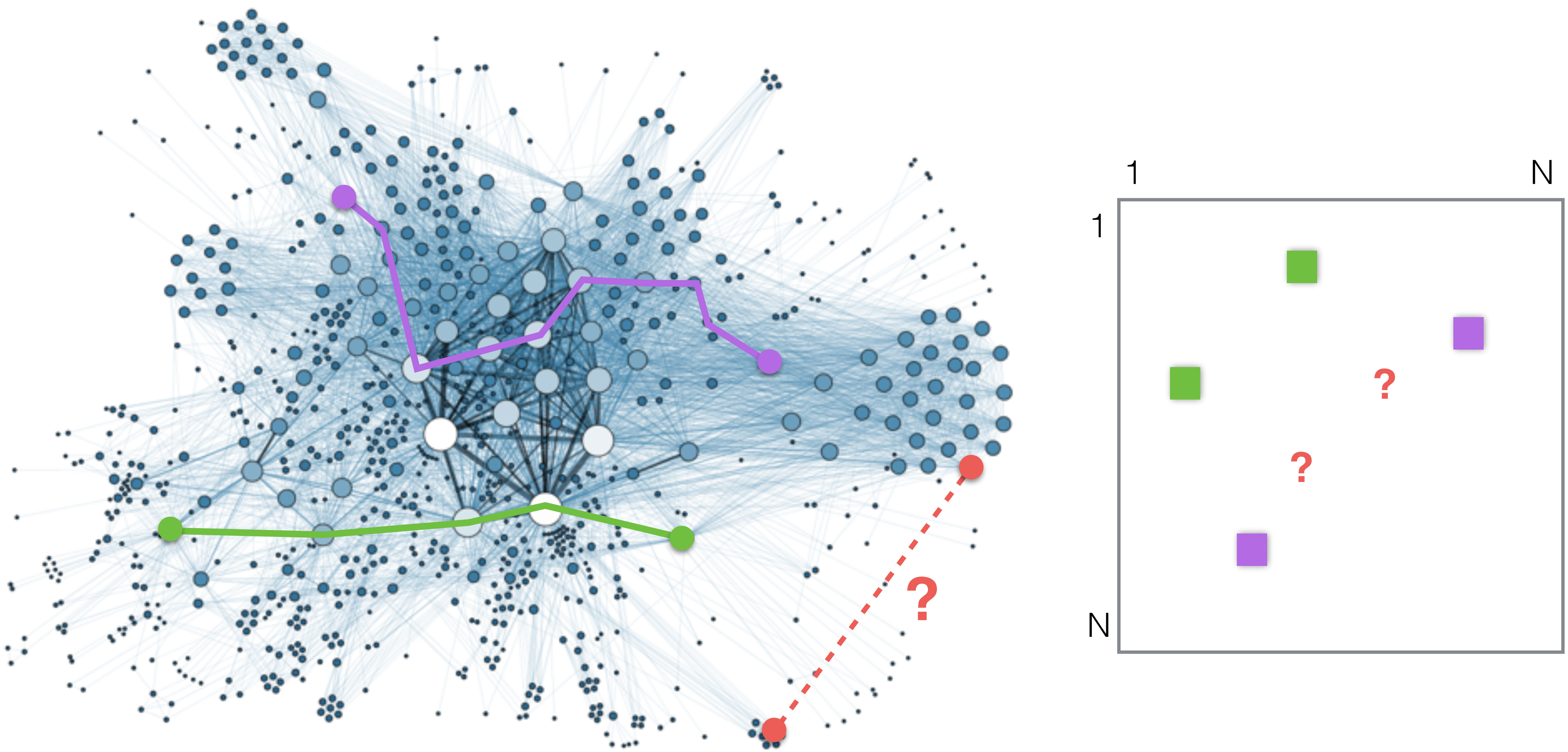

Multiple subspaces

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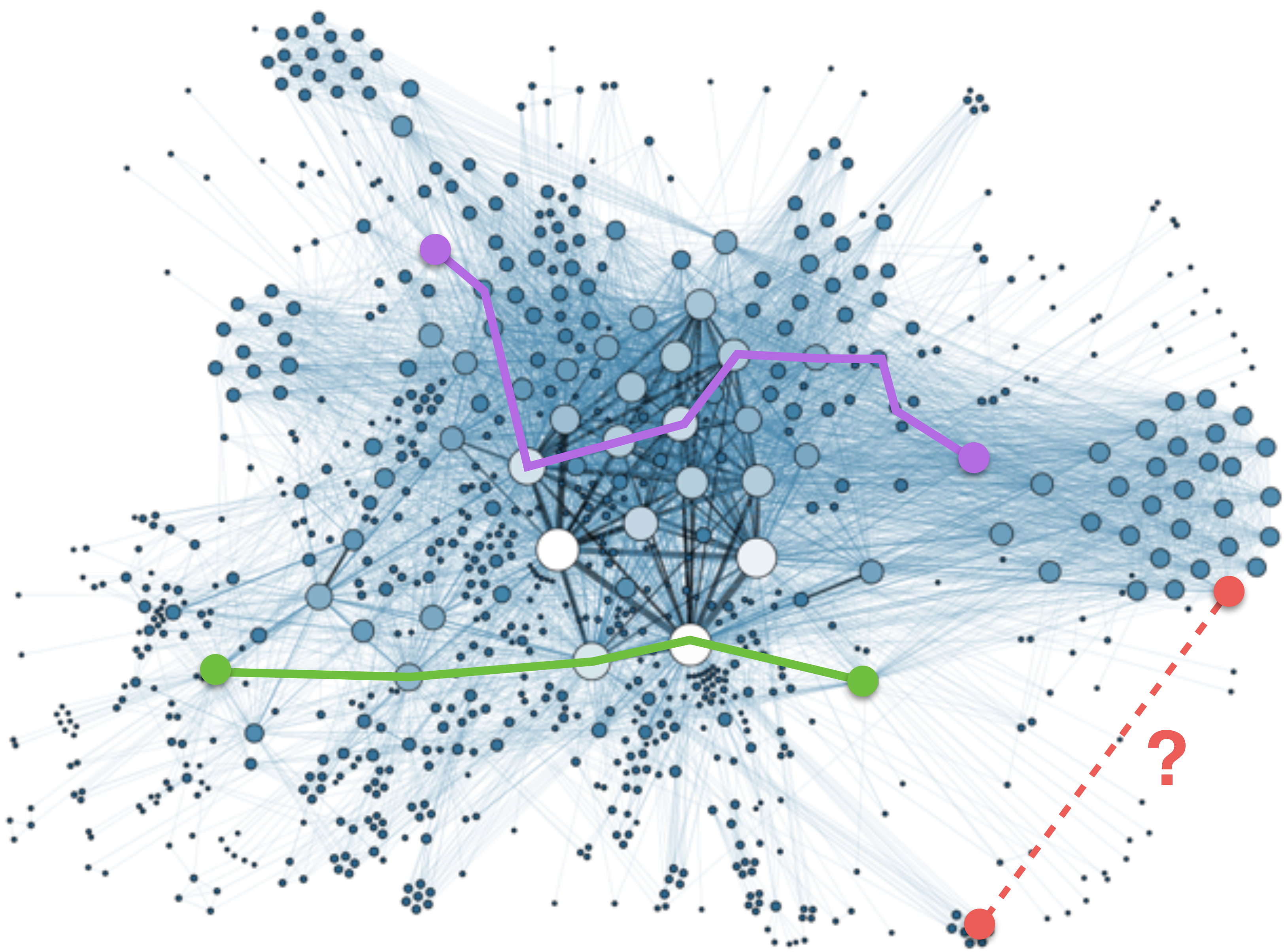
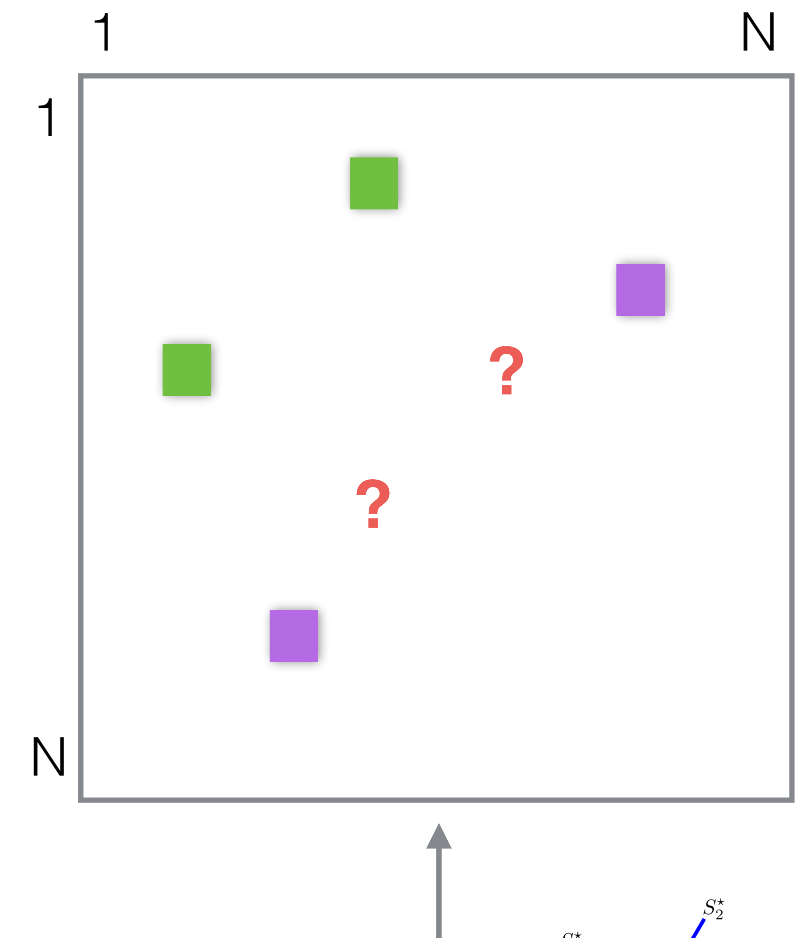
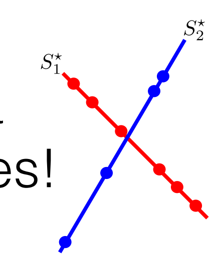




Columns lie in a
union of subspaces!



Multiple subspaces

We want to find them all!

Multiple subspaces

We want to find them all!

Multiple subspaces

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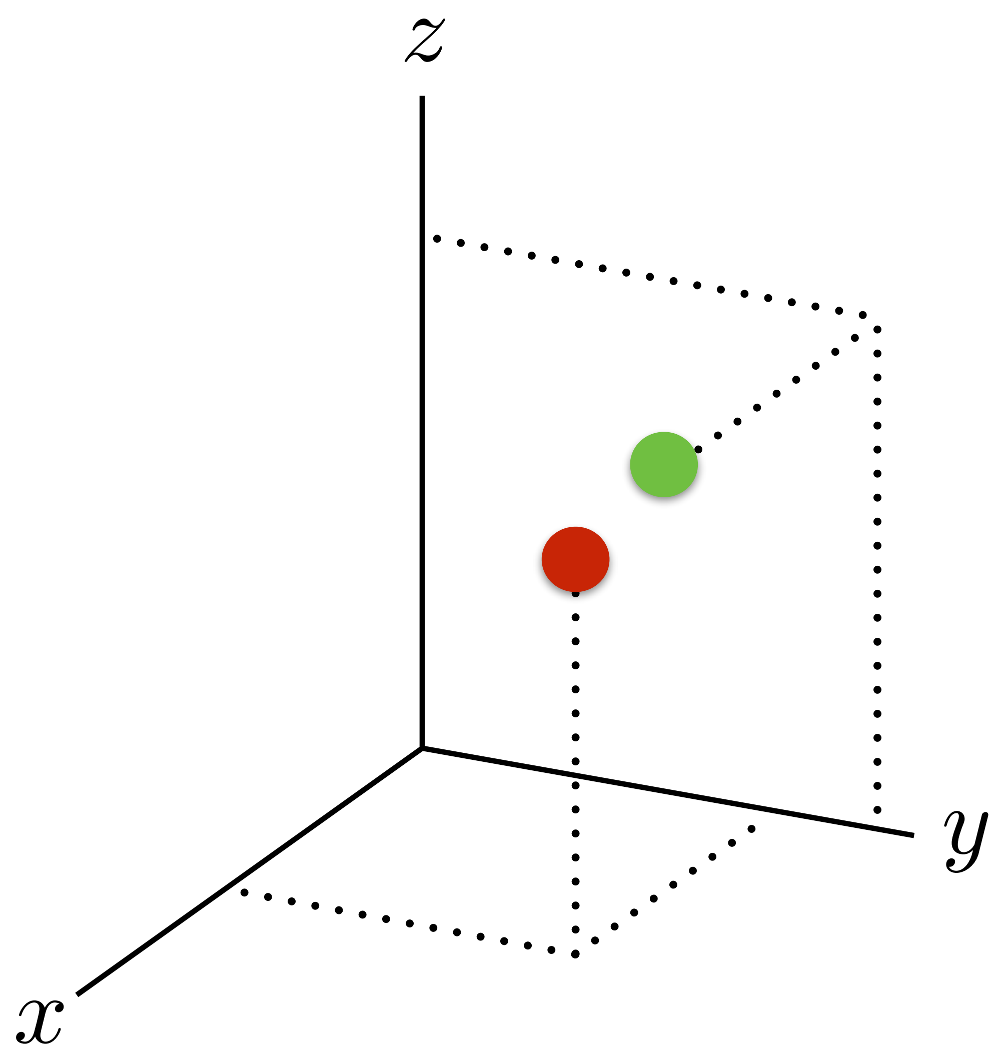
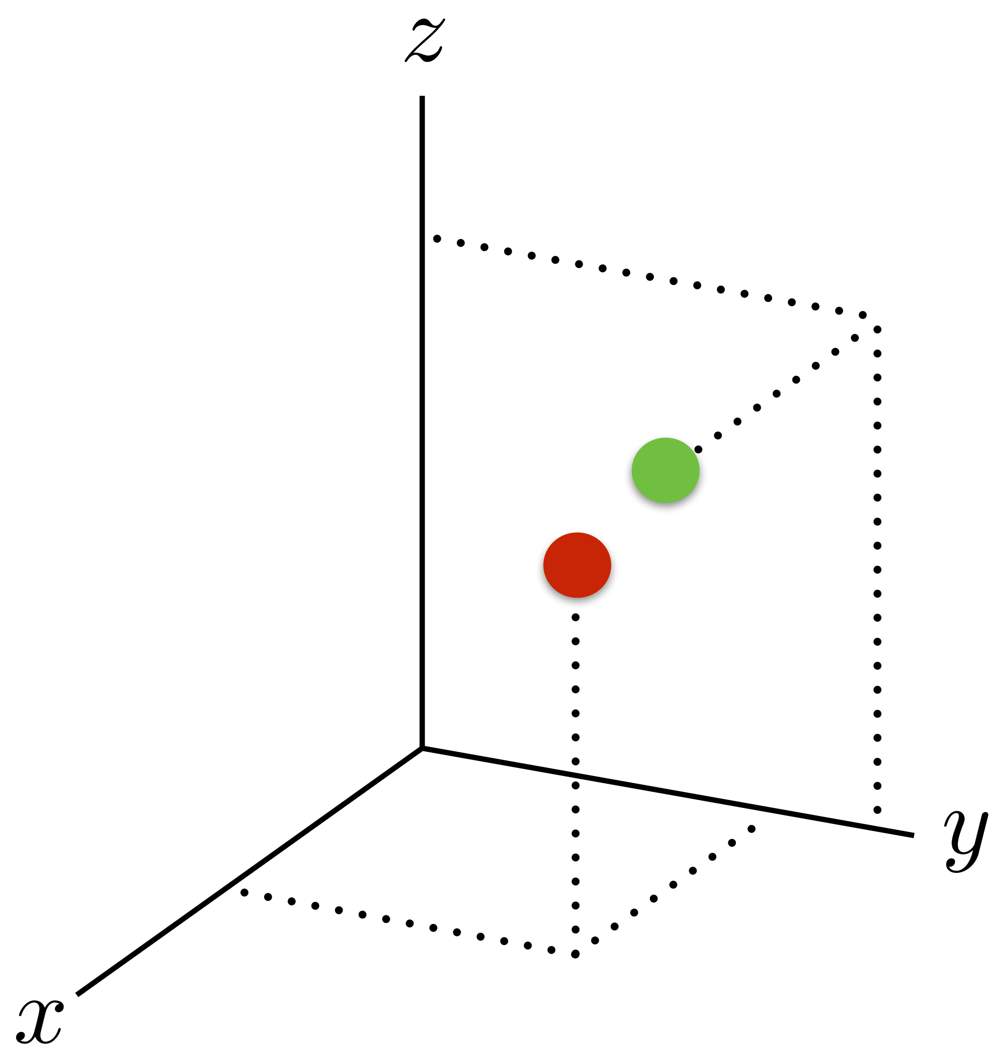


$$\begin{bmatrix} x_1 & \cdot \\ y_1 & y_2 \\ \cdot & z_2 \end{bmatrix}$$

Things get more complicated

- We don’t know where points are :(
- We don’t know which go together :(



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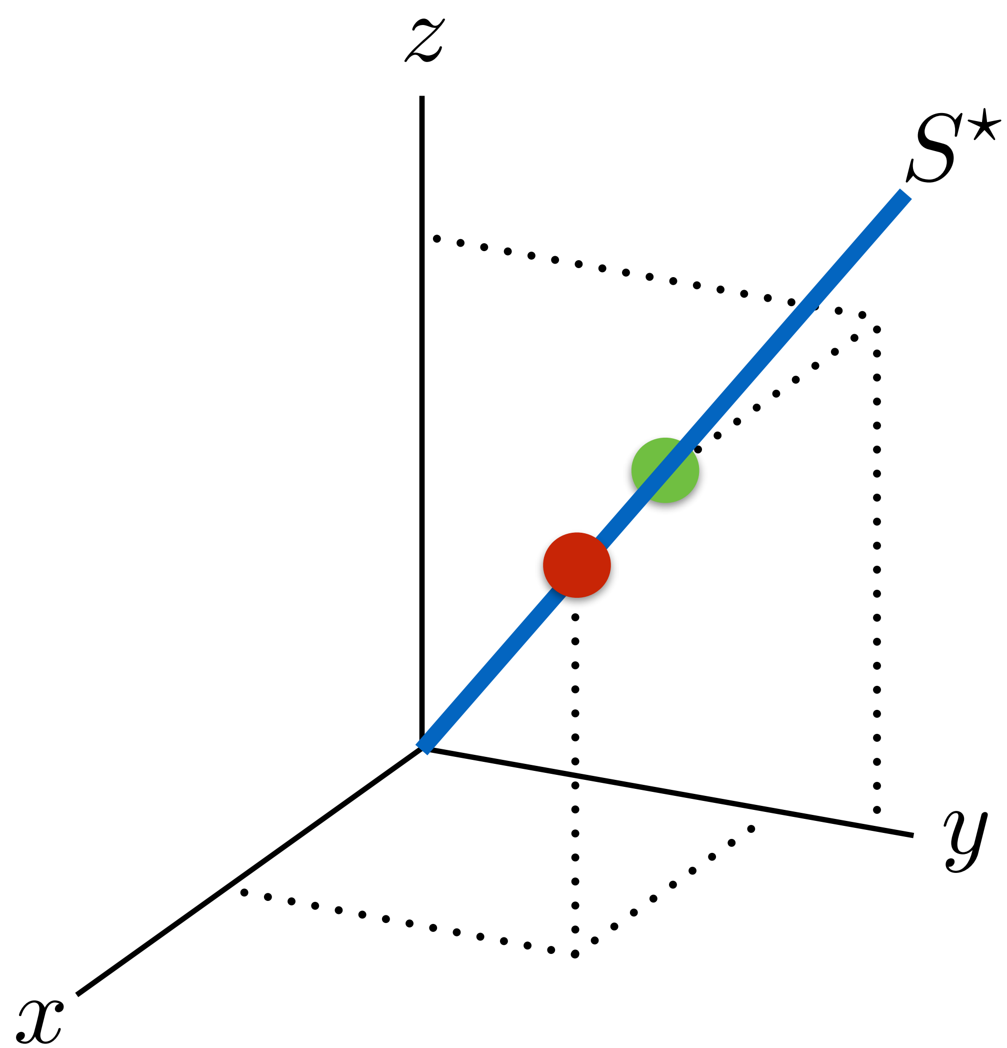

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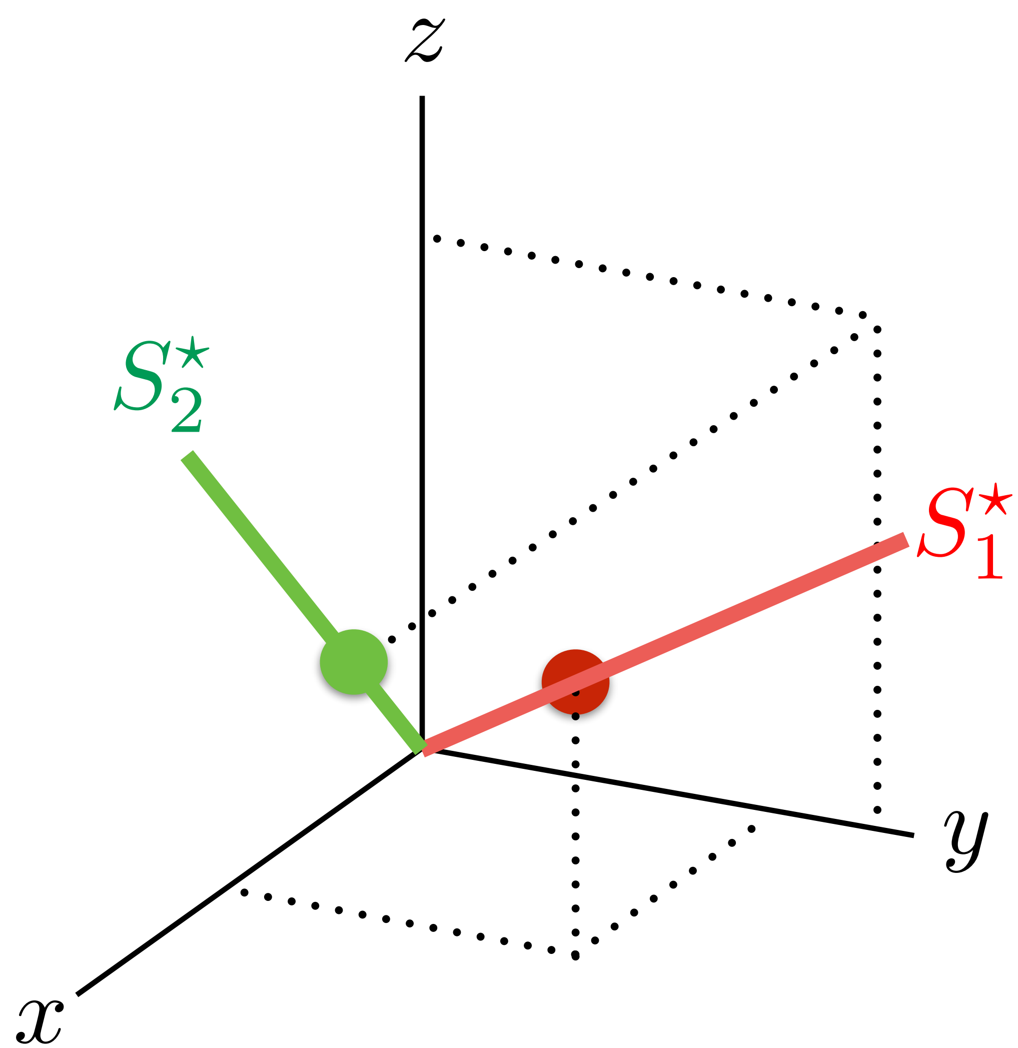

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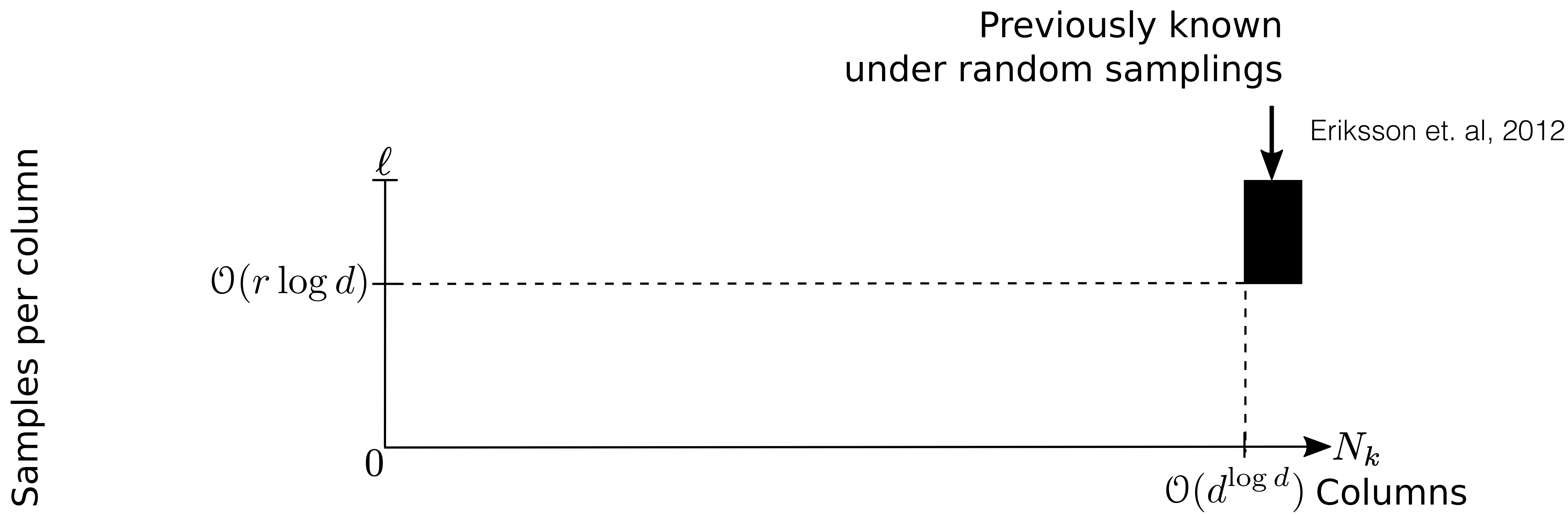
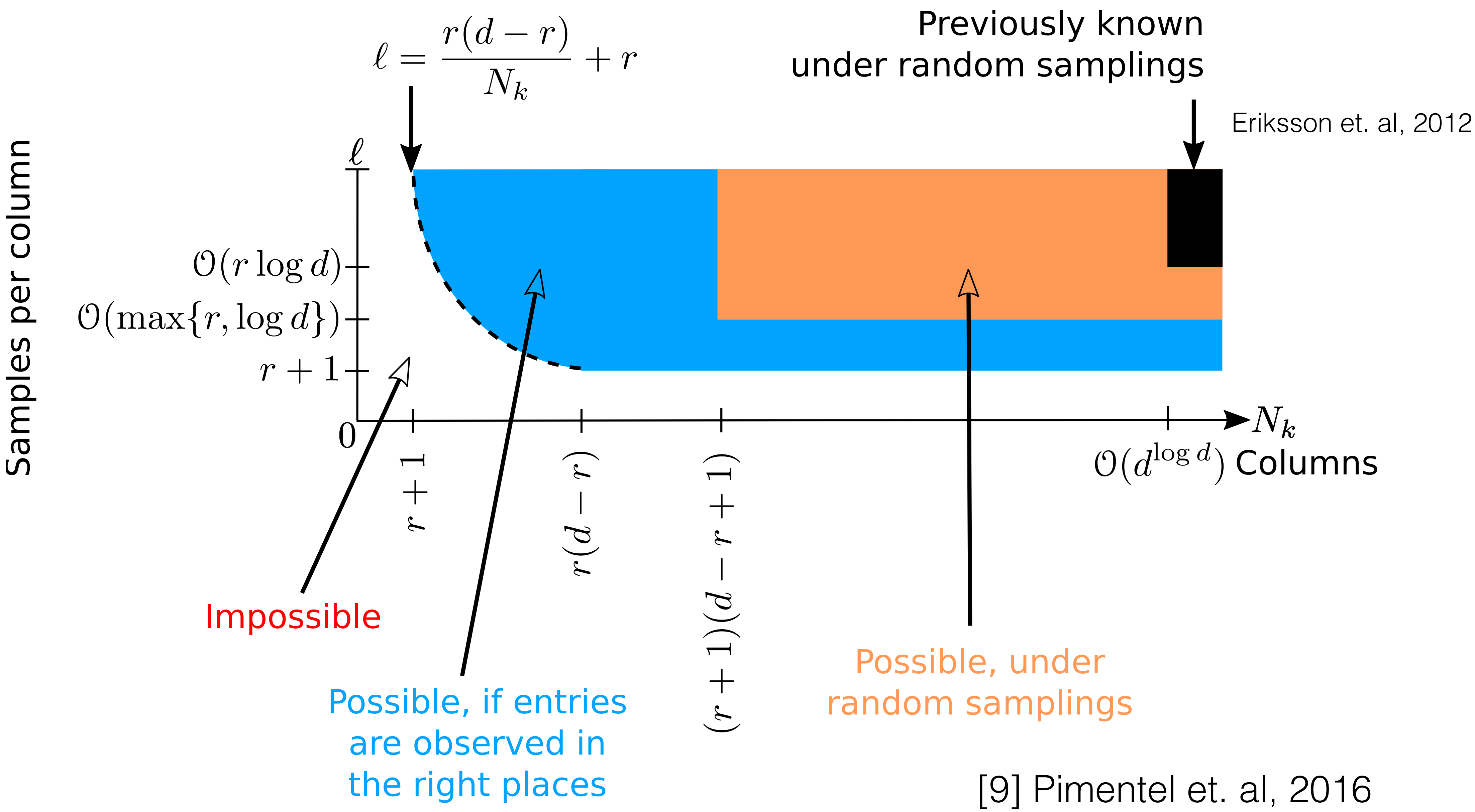
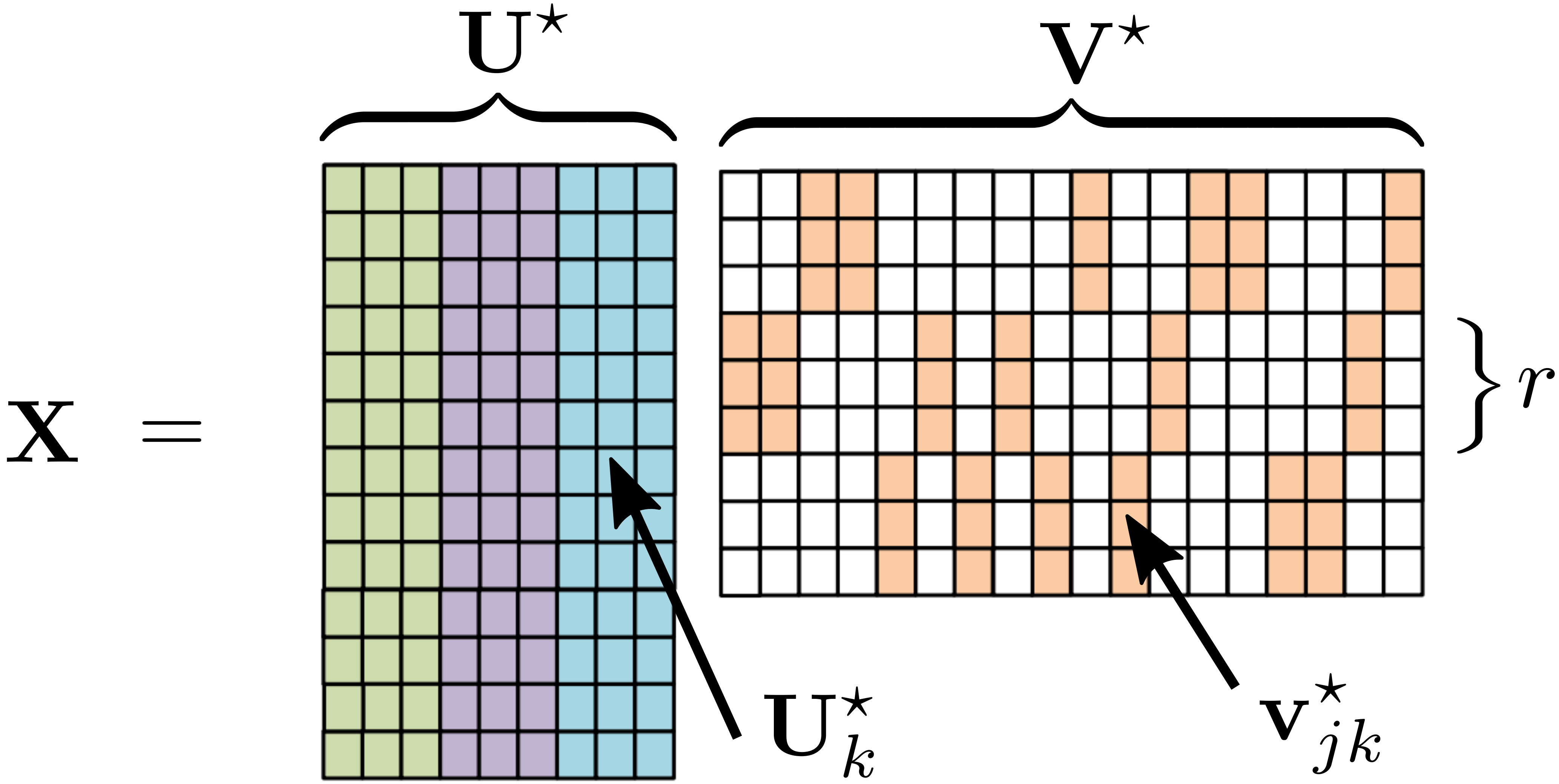


Information-theoretic requirements

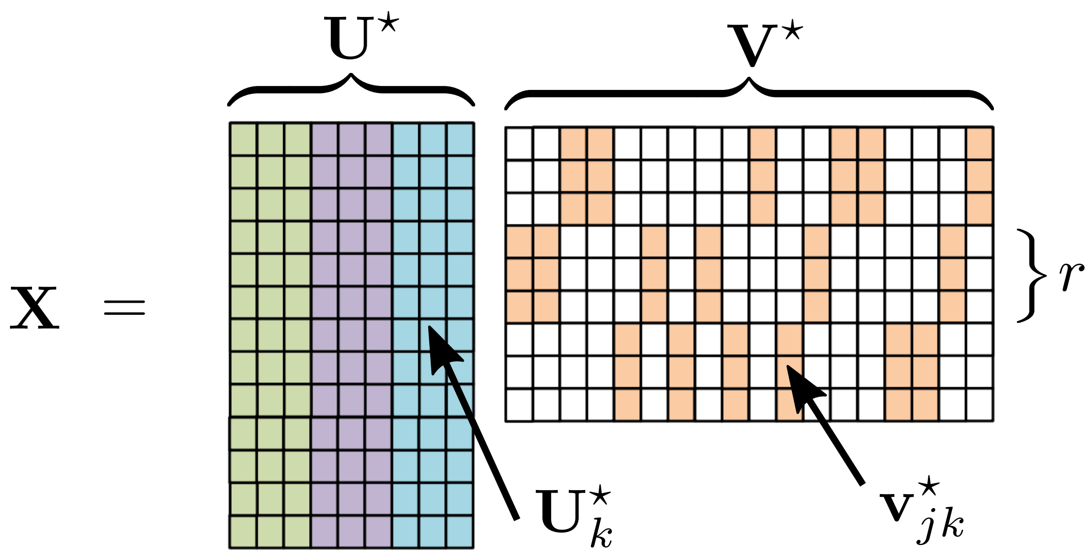


Information-theoretic requirements



State-of-the-art Algorithms

[7] Pimentel et. al, 2016



Algorithm 1: Group-Sparse Subspace Clustering

Input: $\mathbf{X}_{\mathbf{\Omega}}, K, r, \lambda$.

Initialize $\widehat{\mathbf{U}} \in \mathbb{R}^{d \times Kr}$ (e.g., using SSC-EWZF).

repeat

$$\widehat{\mathbf{V}} = \arg\min_{\mathbf{V}} \|\mathbf{\Omega}(\mathbf{X} - \widehat{\mathbf{U}}\mathbf{V})\|_F^2 + \lambda \sum_{j,k=1}^{N,K} \|\mathbf{v}_{jk}\|_2.$$

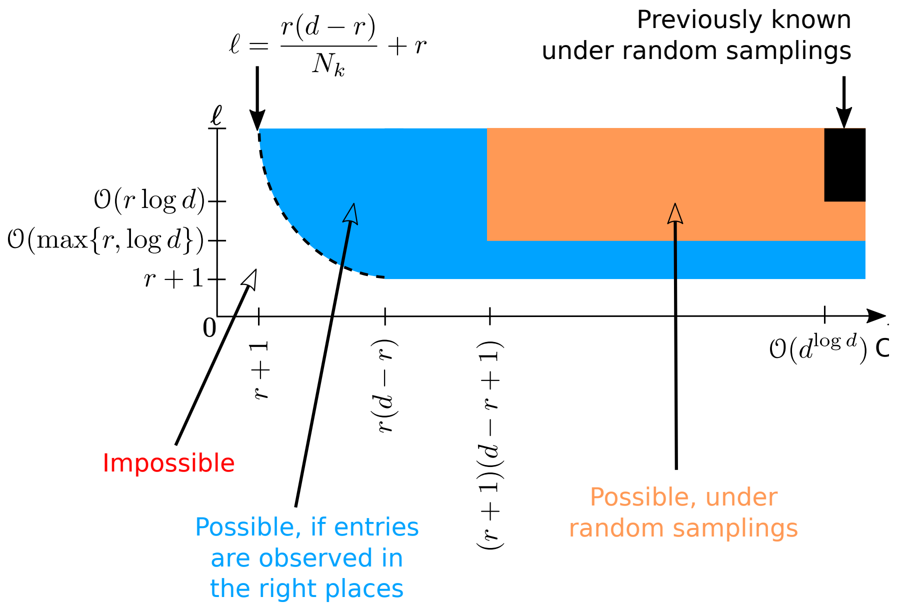
$$\widehat{\mathbf{U}} = \arg\min_{\mathbf{U}\,:\,\|\mathbf{U}\|_F \leq 1} \|\mathbf{\Omega}(\mathbf{X} - \mathbf{U}\widehat{\mathbf{V}})\|_F.$$

until convergence;

Output: $\widehat{\mathbf{U}}, \widehat{\mathbf{V}}$.

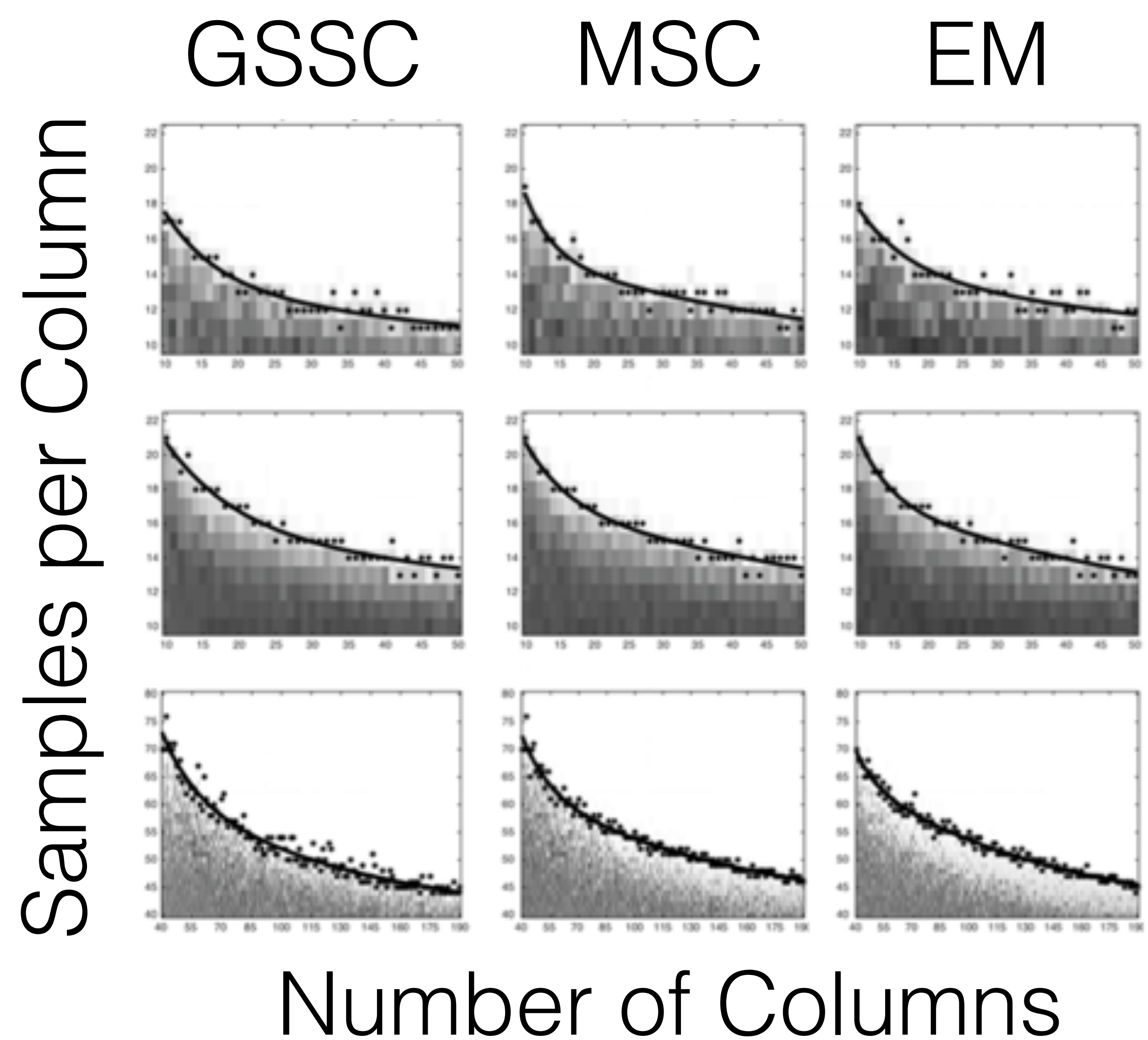
State-of-the-art Algorithms

[7] Pimentel et. al, 2016



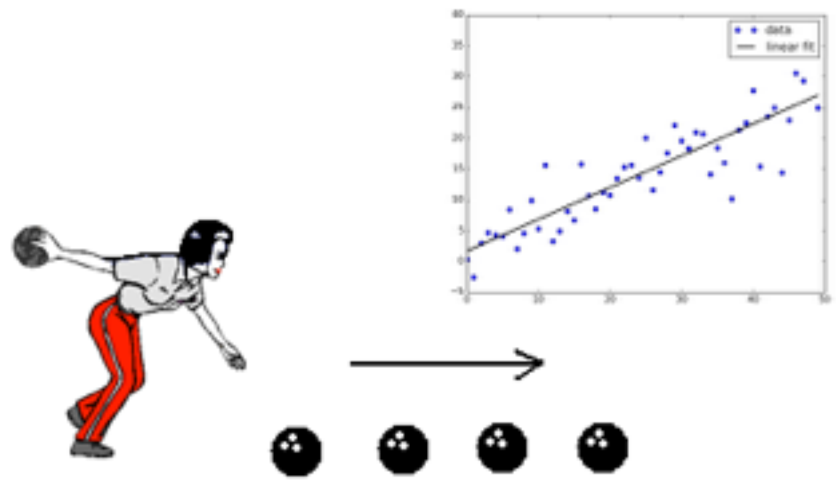
[9] Pimentel et. al, 2016

[7] Pimentel et. al, 2016



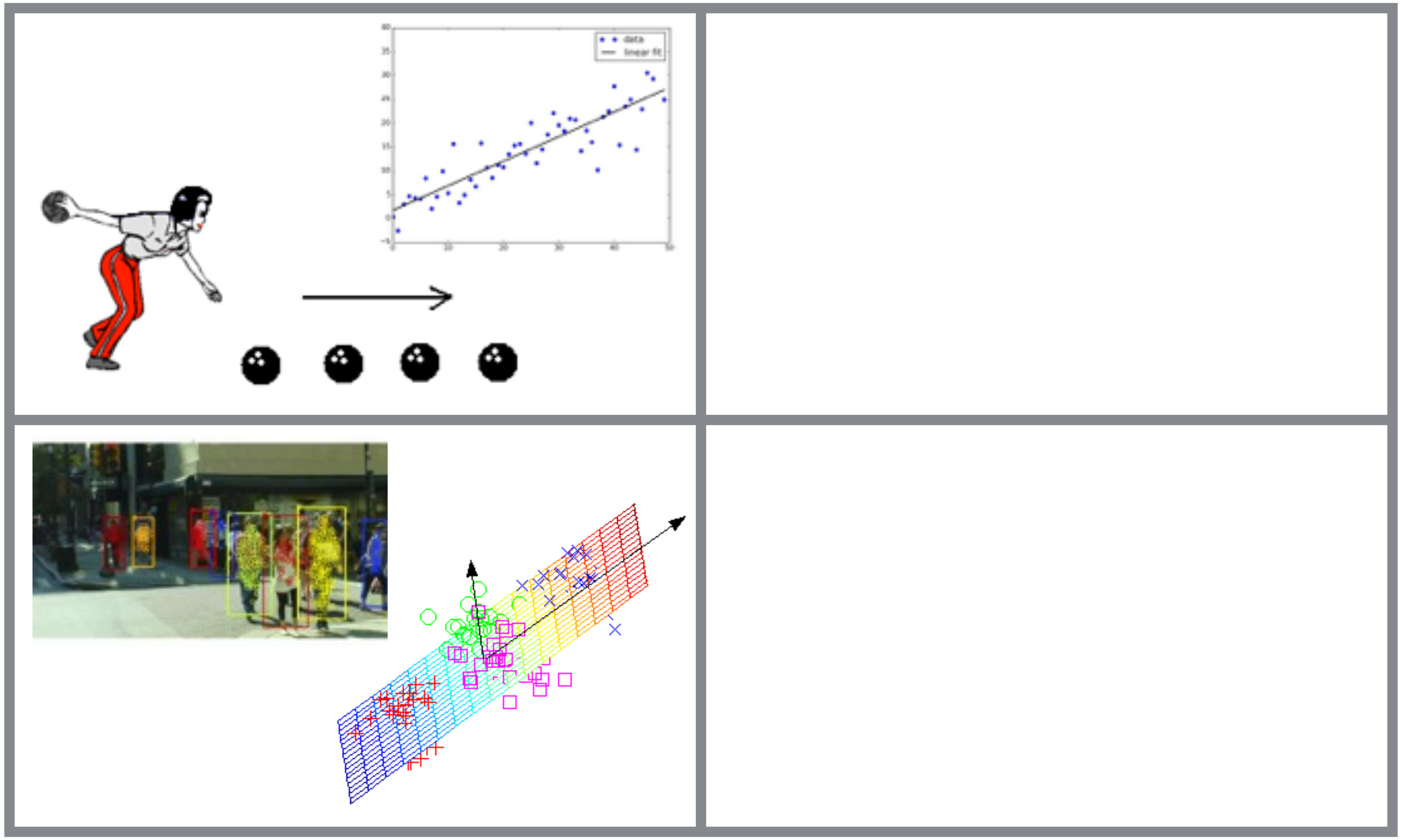
Theory matches Practice

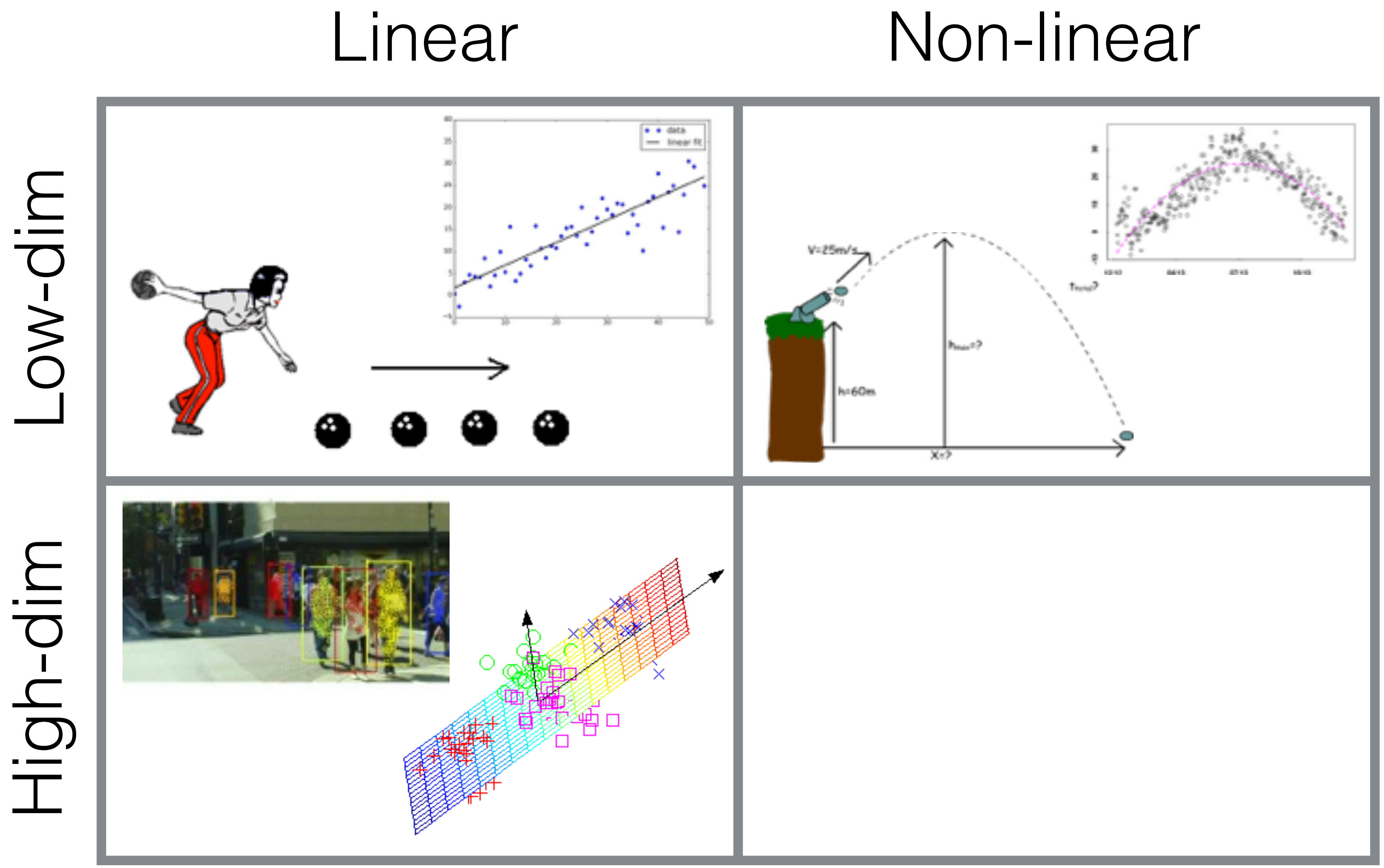

NEXT STEPS

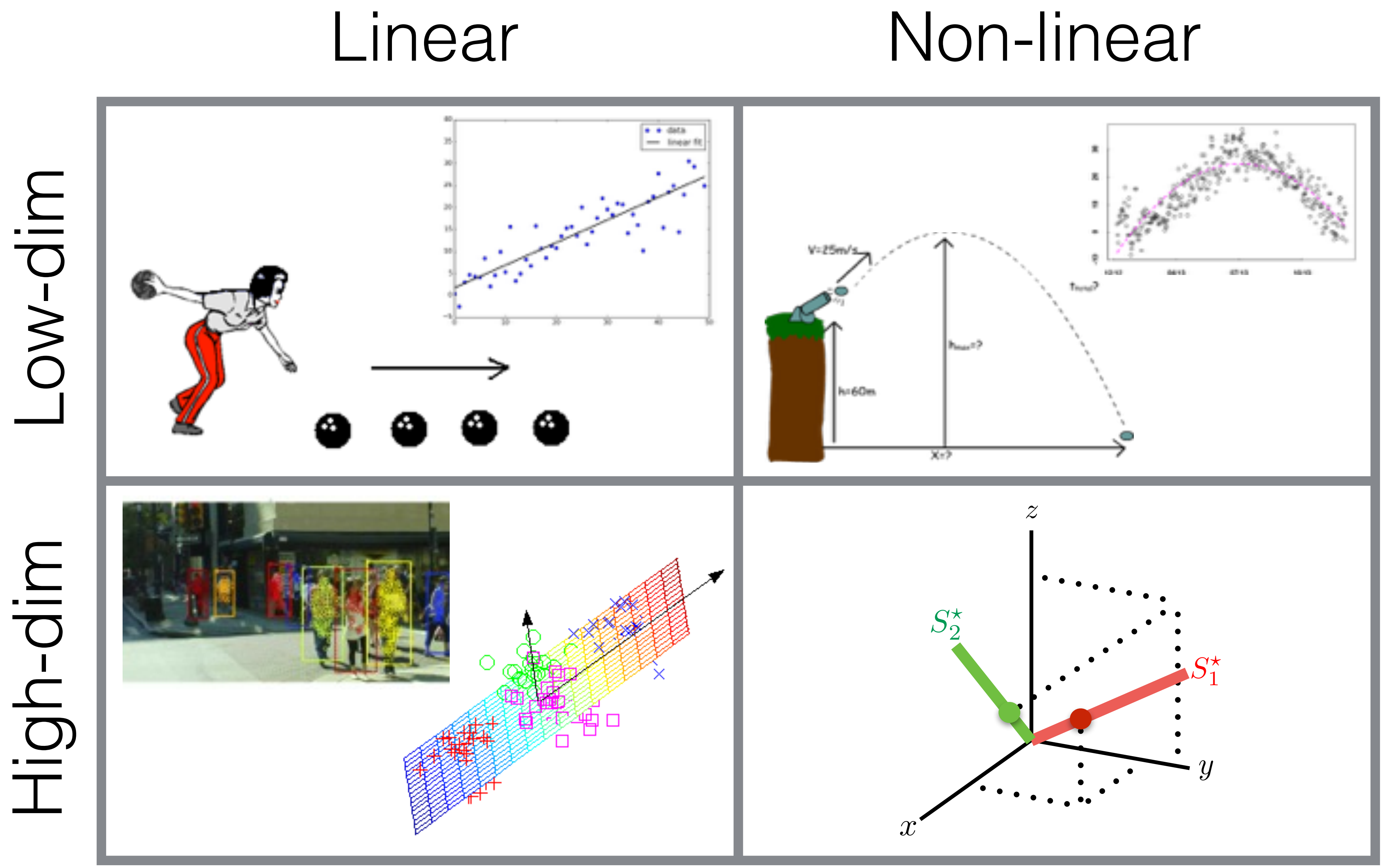

data
linear fit

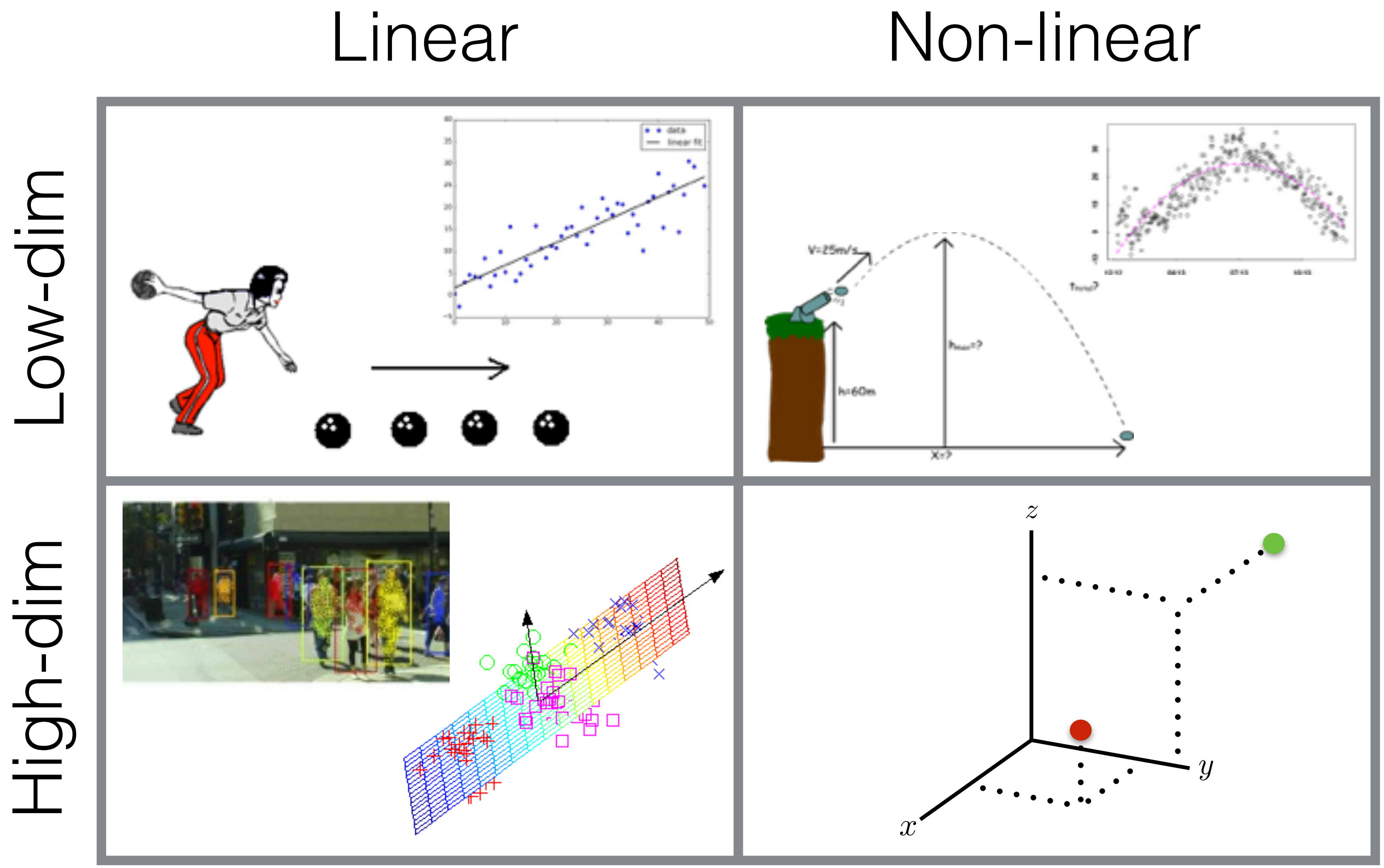
Low-dim

High-dim




Linear
Non-linear
Low-dim
High-dim
data
linear fit
V=25m/s
h=60m
X=?


Linear
Non-linear
Low-dim
High-dim
V=25m/s
h=60m
z
$S_2^\star$
$S_1^\star$
x
y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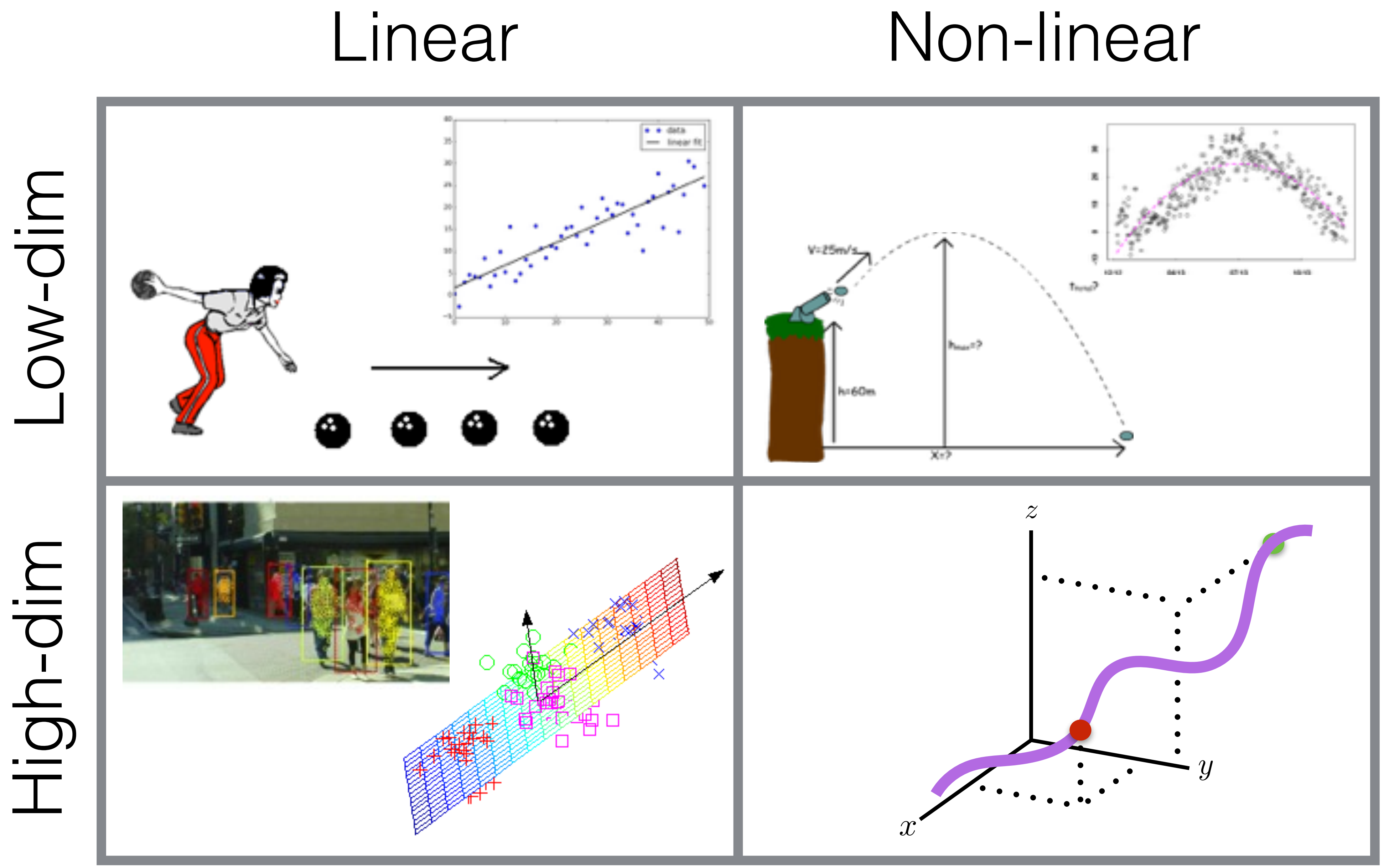
Linear
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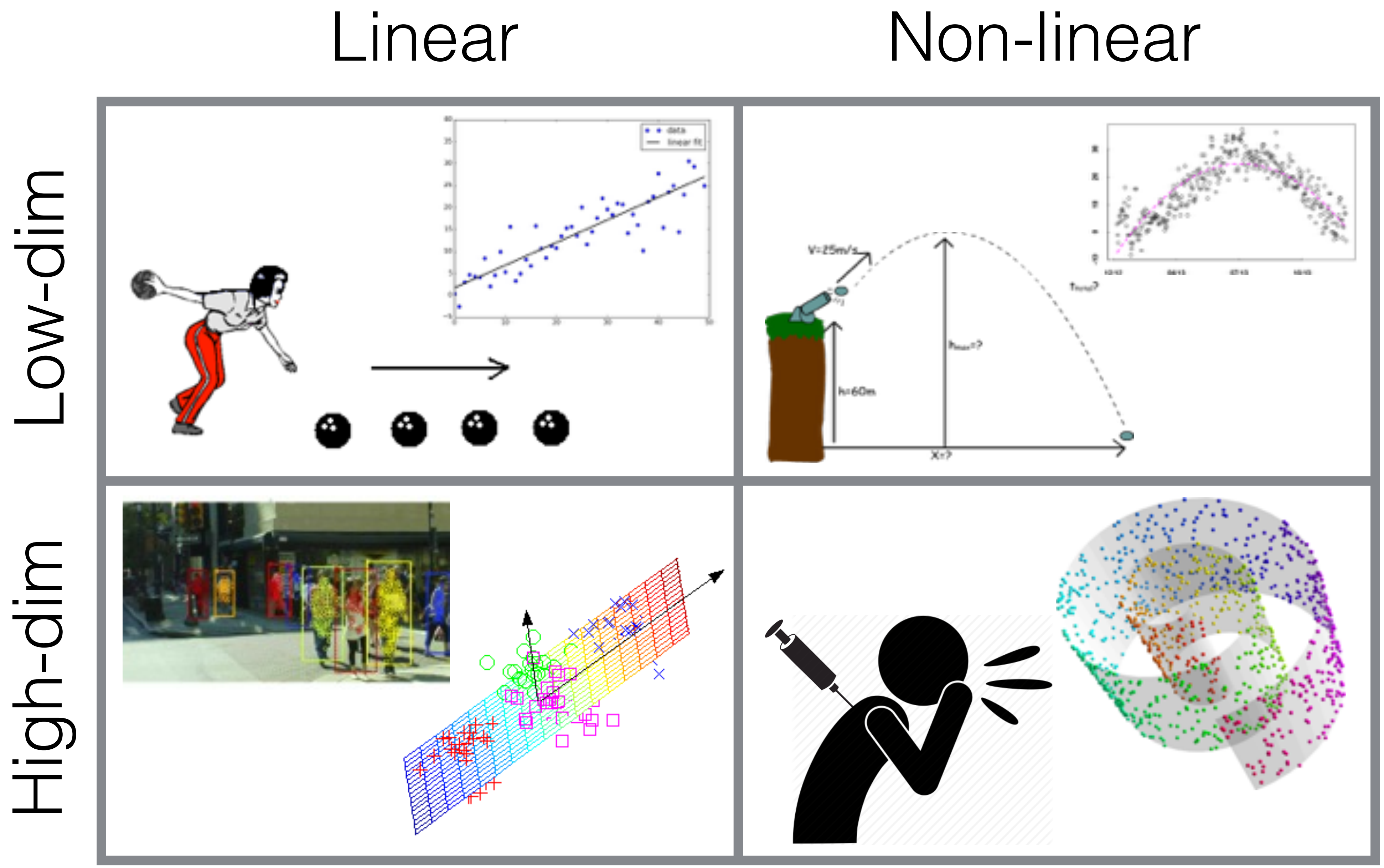
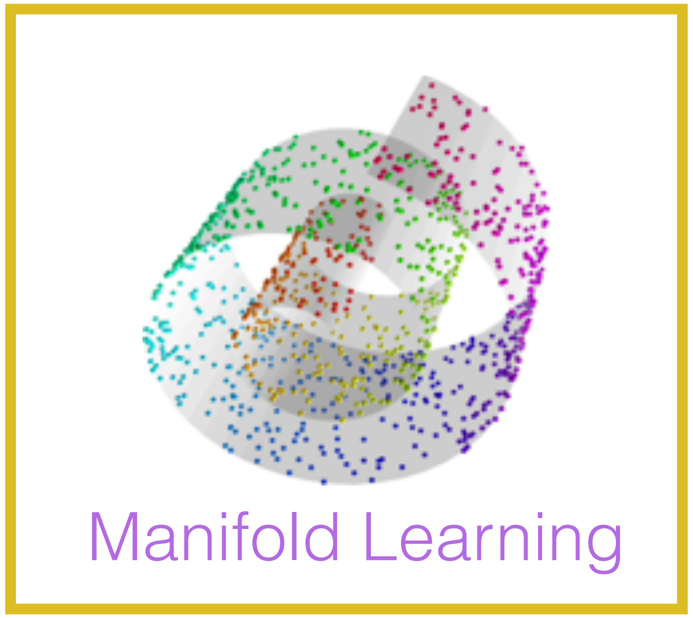
h=60m
z
y
x



Learning Manifolds

(Algebraic Varieties)


NEXT STEPS


Manifold Learning


NEXT STEPS

De Hoop

Manifold Learning

De Hoop

Manifold Learning

Samples \ Computation	Good	Bad
Good		
Bad		Who Cares

SCMD Algorithms

De Hoop

Manifold Learning

Computation

Samples

	HRMC Eriksson et. al, 2012	Who Cares

SCMD Algorithms

De Hoop

Manifold Learning

Computation

Samples	Computation 👍	Computation 👎
👍		Polynomials [10] Pimentel et. al, 2016
👎	HRMC Eriksson et. al, 2012	Who Cares

SCMD Algorithms

De Hoop

Manifold Learning

Samples \ Computation	👍	👎
👍	• EM [14] Pimentel et. al, 2014 • GSSC [7] Pimentel et. al, 2016 • MSC [7] Pimentel et. al, 2016 • SSC-EWZF Wang et. al, 2016 • K-GROUSE Balzano et. al, 2016	Polynomials [10] Pimentel et. al, 2016
👎	HRMC Eriksson et. al, 2012	Who Cares

SCMD Algorithms

De Hoop

Manifold Learning

	Computation 👍	Computation 👎
Samples 👍	• EM [14] Pimentel et. al, 2014 • GSSC [7] Pimentel et. al, 2016 • MSC [7] Pimentel et. al, 2016 • SSC-EWZF Wang et. al, 2016 • K-GROUSE Balzano et. al, 2016 Provable?	Polynomials [10] Pimentel et. al, 2016
Samples 👎	HRMC Eriksson et. al, 2012	Who Cares

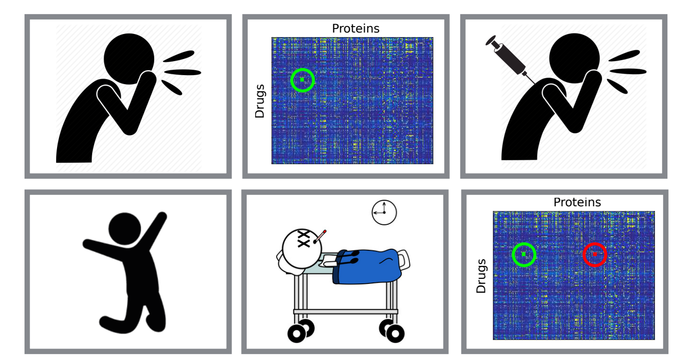
SCMD Algorithms

De Hoop

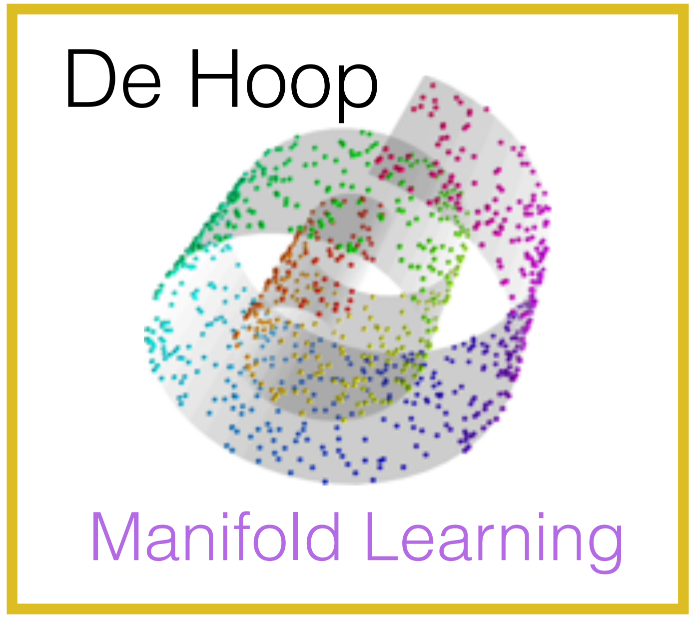
Manifold Learning

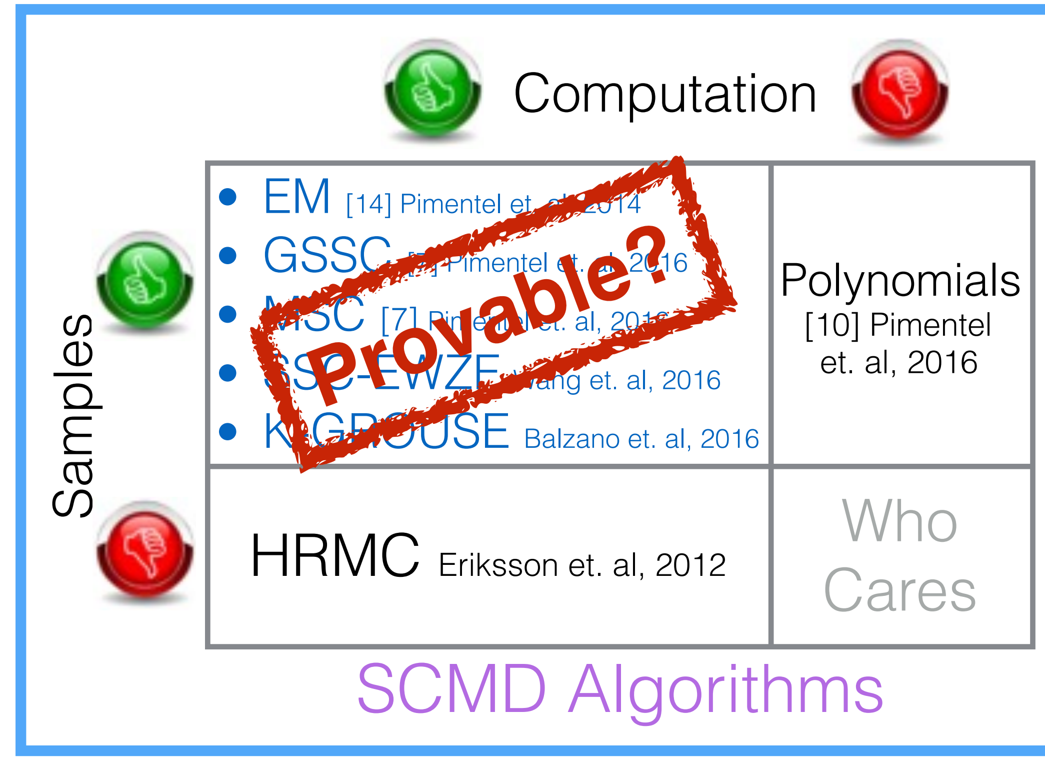
Samples \ Computation	👍	👎
👍	• EM [14] Pimentel et. al, 2014 • GSSC [7] Pimentel et. al, 2016 • MSC [7] Pimentel et. al, 2016 • SSC-EWZF Yang et. al, 2016 • K-GROUSE Balzano et. al, 2016 **Provable?**	Polynomials [10] Pimentel et. al, 2016
👎	HRMC Eriksson et. al, 2012	Who Cares

SCMD Algorithms

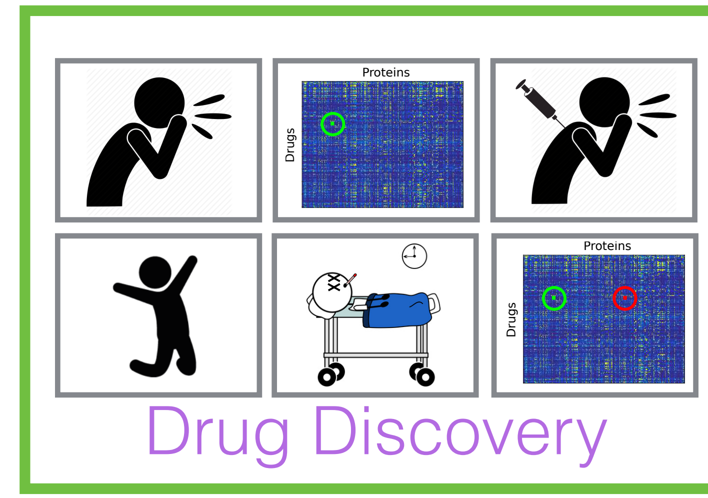


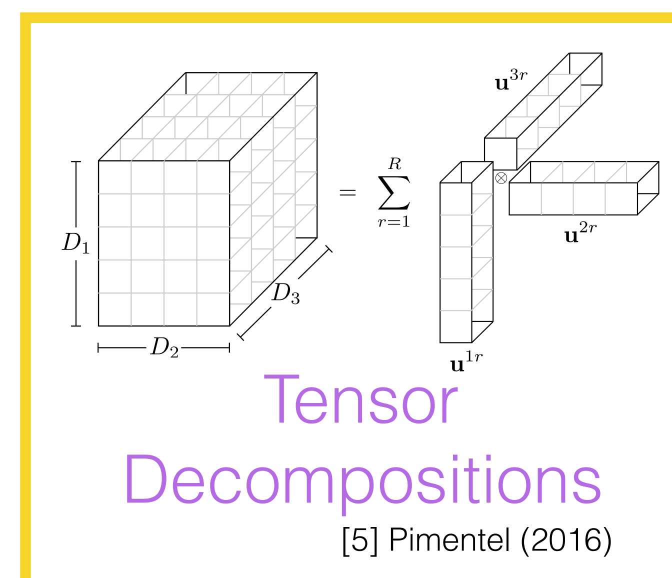
Drug Discovery


De Hoop
Manifold Learning


Computation
Samples
EM [14] Pimentel et. al, 2014
GSSC [7] Pimentel et. al, 2016
MSC [7] Pimentel et. al, 2016
SSC-EWZF Yang et. al, 2016
K-GROUSE Balzano et. al, 2016
Provable?
Polynomials
[10] Pimentel
et. al, 2016
HRMC Eriksson et. al, 2012
Who
Cares
SCMD Algorithms


NEXT STEPS


Proteins
Drugs
Proteins
Drugs
Drug Discovery

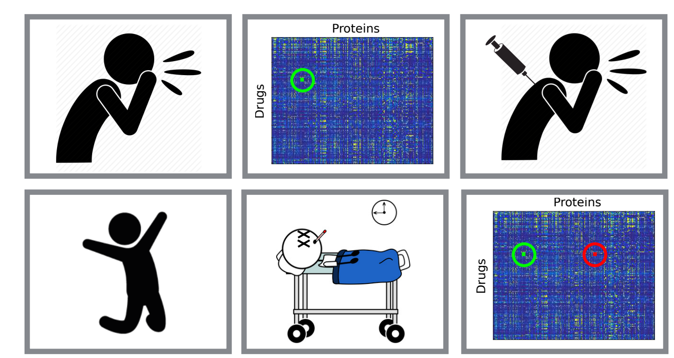

D_1
D_2
D_3
$= \sum_{r=1}^{R}$
$\mathbf{u}^{3r}$
$\mathbf{u}^{2r}$
$\mathbf{u}^{1r}$
Tensor
Decompositions
[5] Pimentel (2016)

De Hoop

Manifold Learning

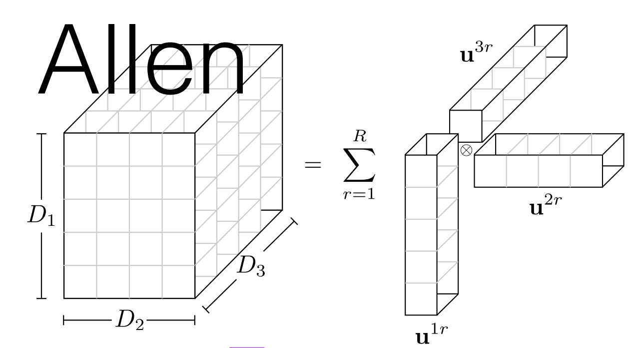
	Computation: good	Computation: bad
Samples: good	• EM [14] Pimentel et. al, 2014 • GSSC Pimentel et. al, 2016 • MSC [7] Pimentel et. al, 2016 • SSC-EWZF Yang et. al, 2016 • K-GROUSE Balzano et. al, 2016 (stamped: Provable?)	Polynomials [10] Pimentel et. al, 2016
Samples: bad	HRMC Eriksson et. al, 2012	Who Cares

SCMD Algorithms



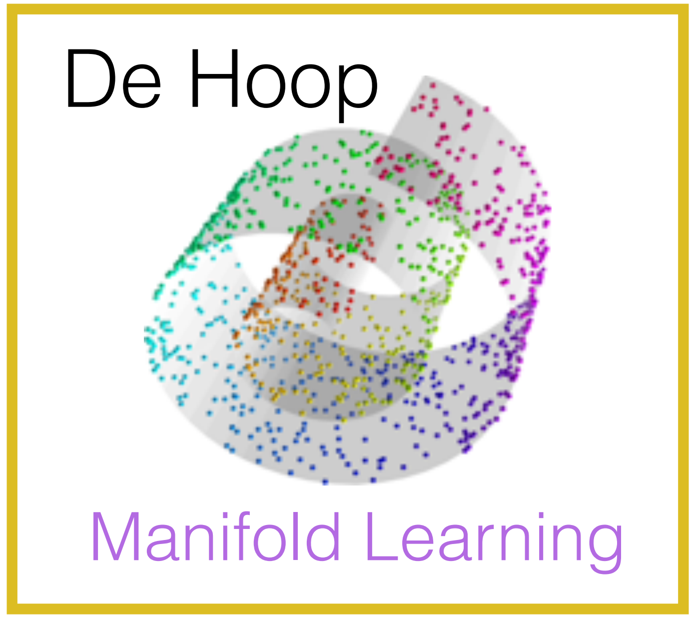
Drug Discovery

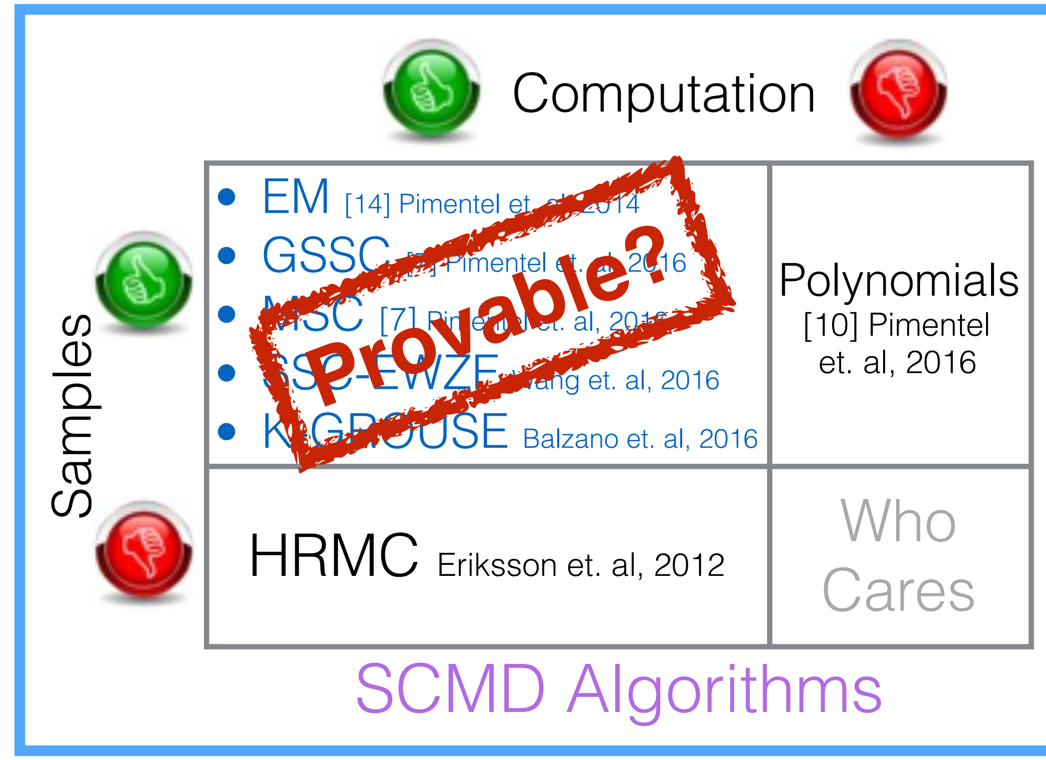
Allen



Tensor Decompositions

[5] Pimentel (2016)


De Hoop
Manifold Learning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Provable?
Polynomials
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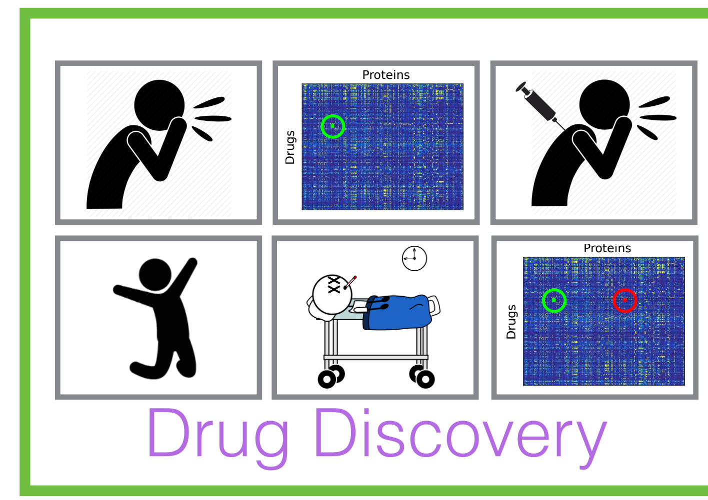
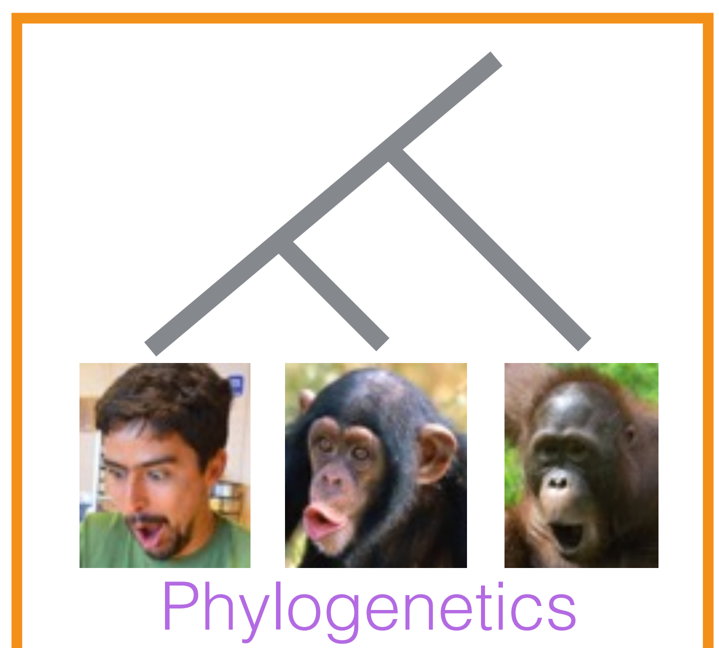
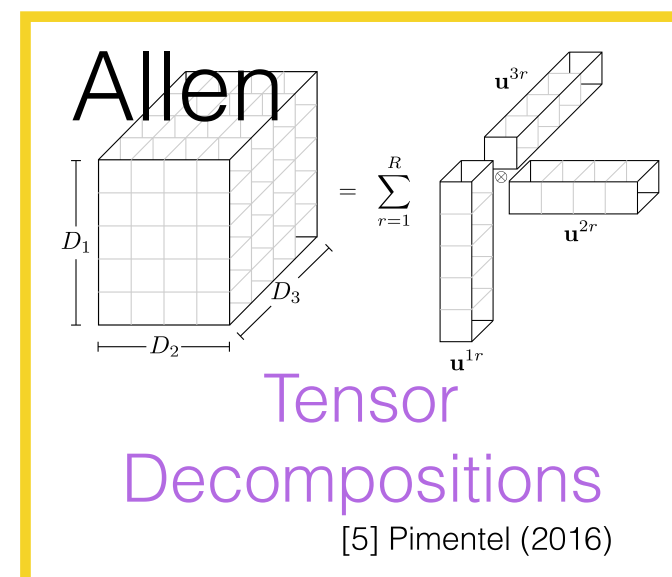
Who
Cares
SCMD Algorithms


NEXT STEPS


Proteins
Drugs
Proteins
Drugs
Drug Discovery


Phylogenetics


Allen
D_1
D_2
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$= \sum_{r=1}^{R}$
$\mathbf{u}^{1r}$
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$\mathbf{u}^{3r}$
Tensor
Decompositions
[5] Pimentel (2016)

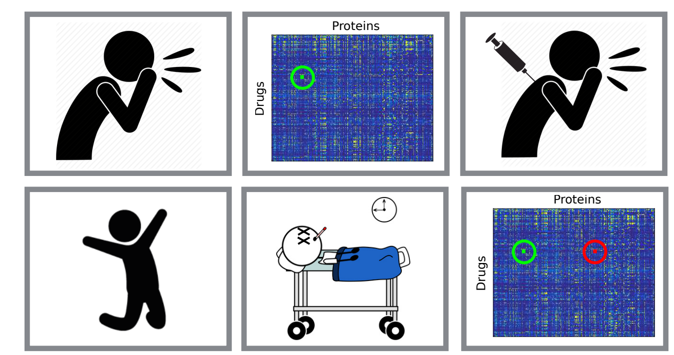
De Hoop

Manifold Learning

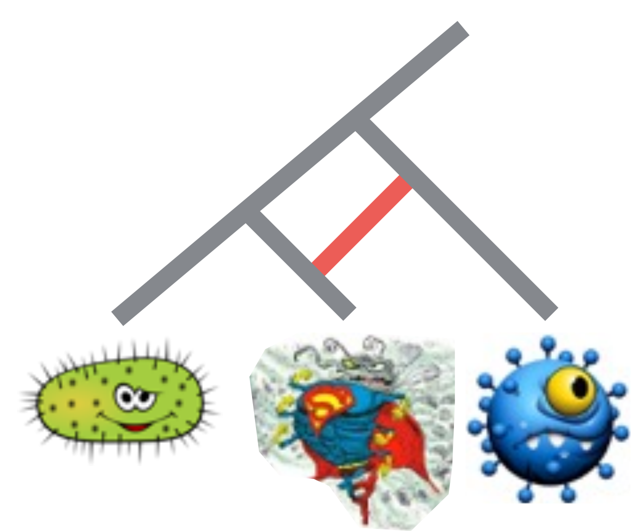
	Computation (thumbs up)	Computation (thumbs down)
Samples (thumbs up)	• EM [14] Pimentel et. al, 2014 • GSSC Pimentel et. al, 2016 • MSC [7] Pimentel et. al, 2016 • SSC-EWZF Yang et. al, 2016 • K-GROUSE Balzano et. al, 2016	Polynomials [10] Pimentel et. al, 2016
Samples (thumbs down)	HRMC Eriksson et. al, 2012	Who Cares

Provable?

SCMD Algorithms

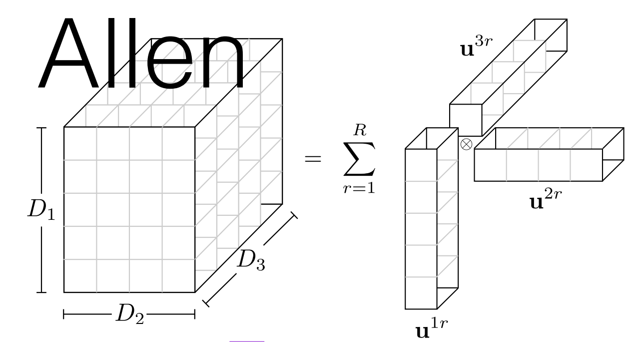


Drug Discovery



Phylogenetics

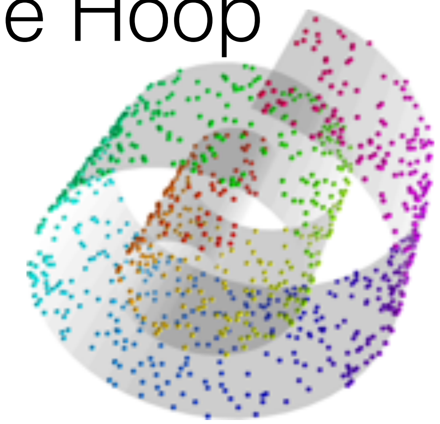
Allen



Tensor Decompositions

[5] Pimentel (2016)

De Hoop



Manifold Learning

Samples \ Computation	👍	👎
👍	• EM [14] Pimentel et. al, 2014 • GSSC Pimentel et. al, 2016 • MSC [7] Pimentel et. al, 2016 • SSC-EWZF Yang et. al, 2016 • K-GROUSE Balzano et. al, 2016 **Provable?**	Polynomials [10] Pimentel et. al, 2016
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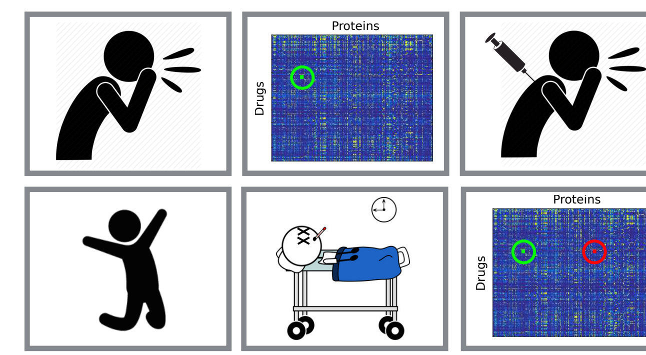
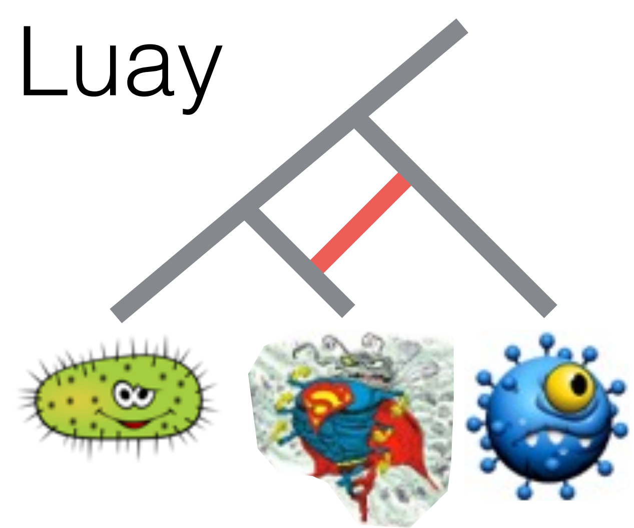
SCMD Algorithms



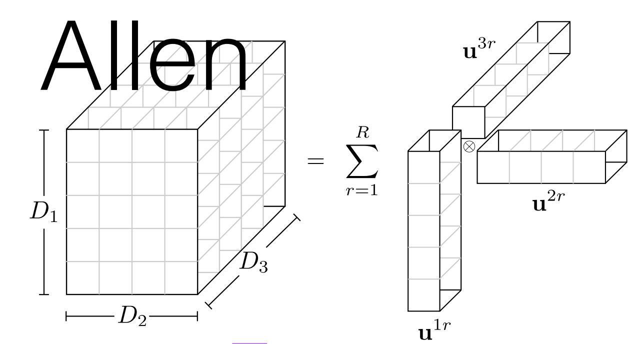
Drug Discovery

Luay



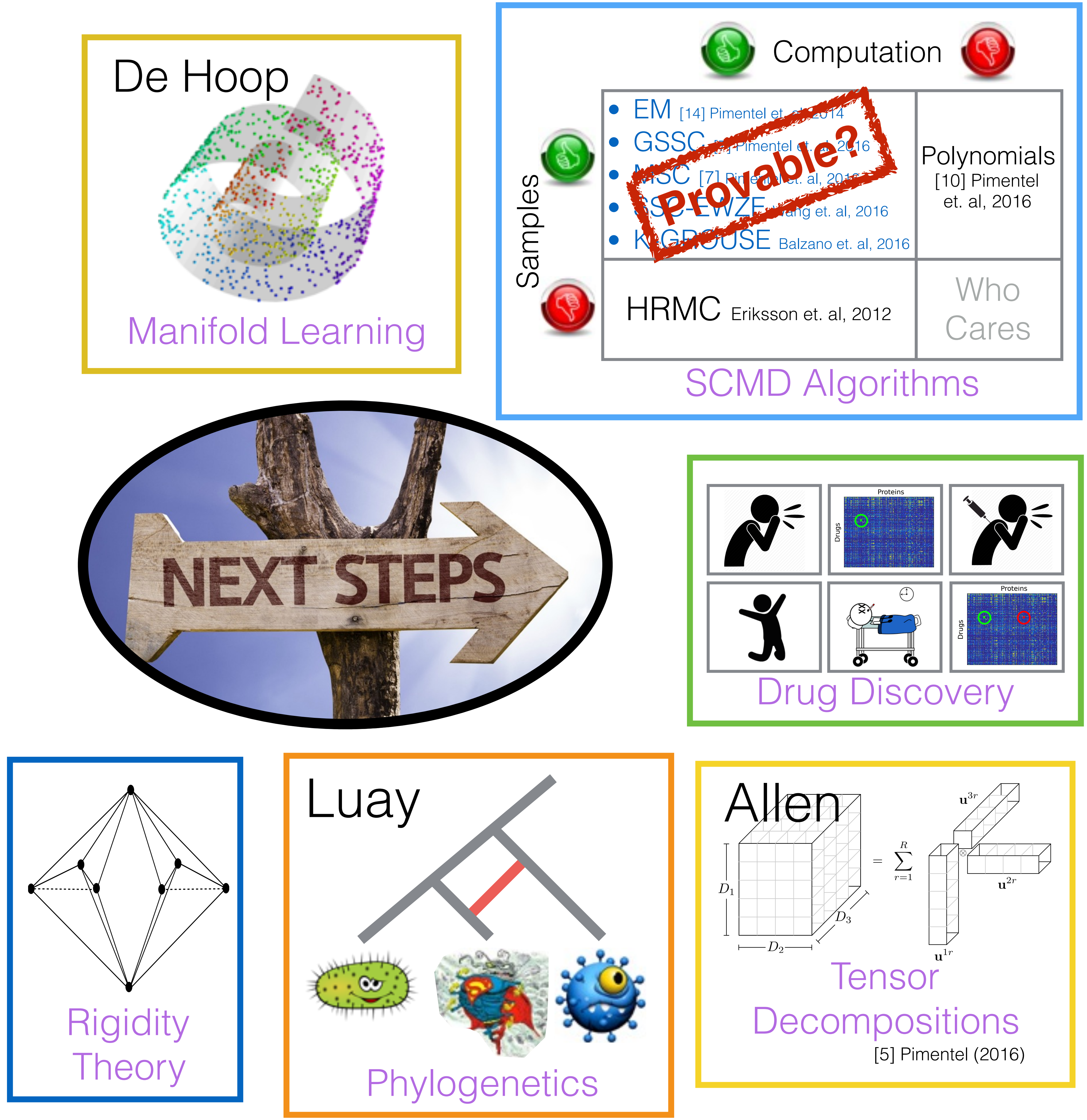
Phylogenetics

Allen



Tensor Decompositions

[5] Pimentel (2016)



De Hoop
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Rigidity Theory
Luay
Phylogenetics
Allen
Tensor Decompositions
[5] Pimentel (2016)

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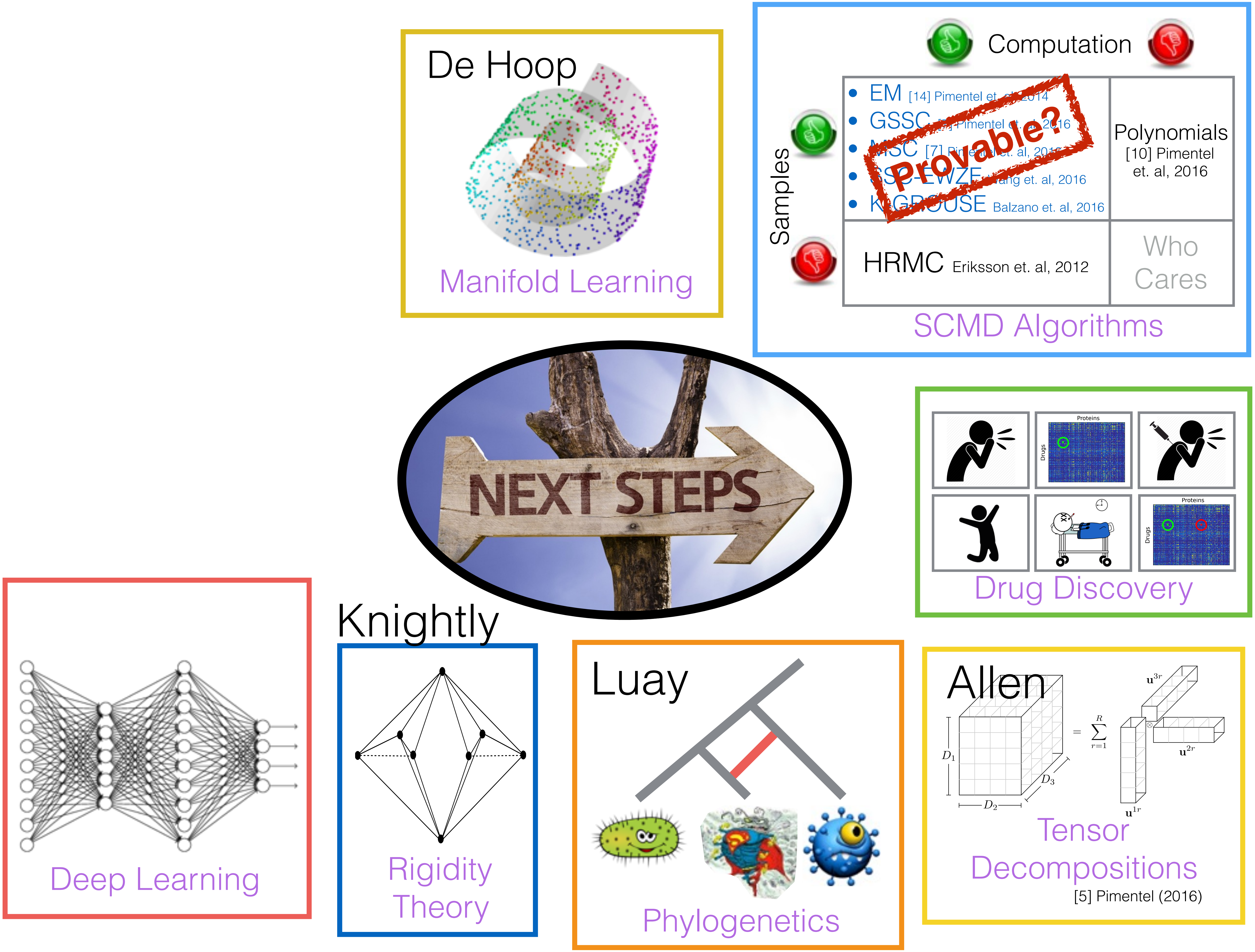
[5] Pimentel (2016)



De Hoop
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EM [14] Pimentel et. al, 2014
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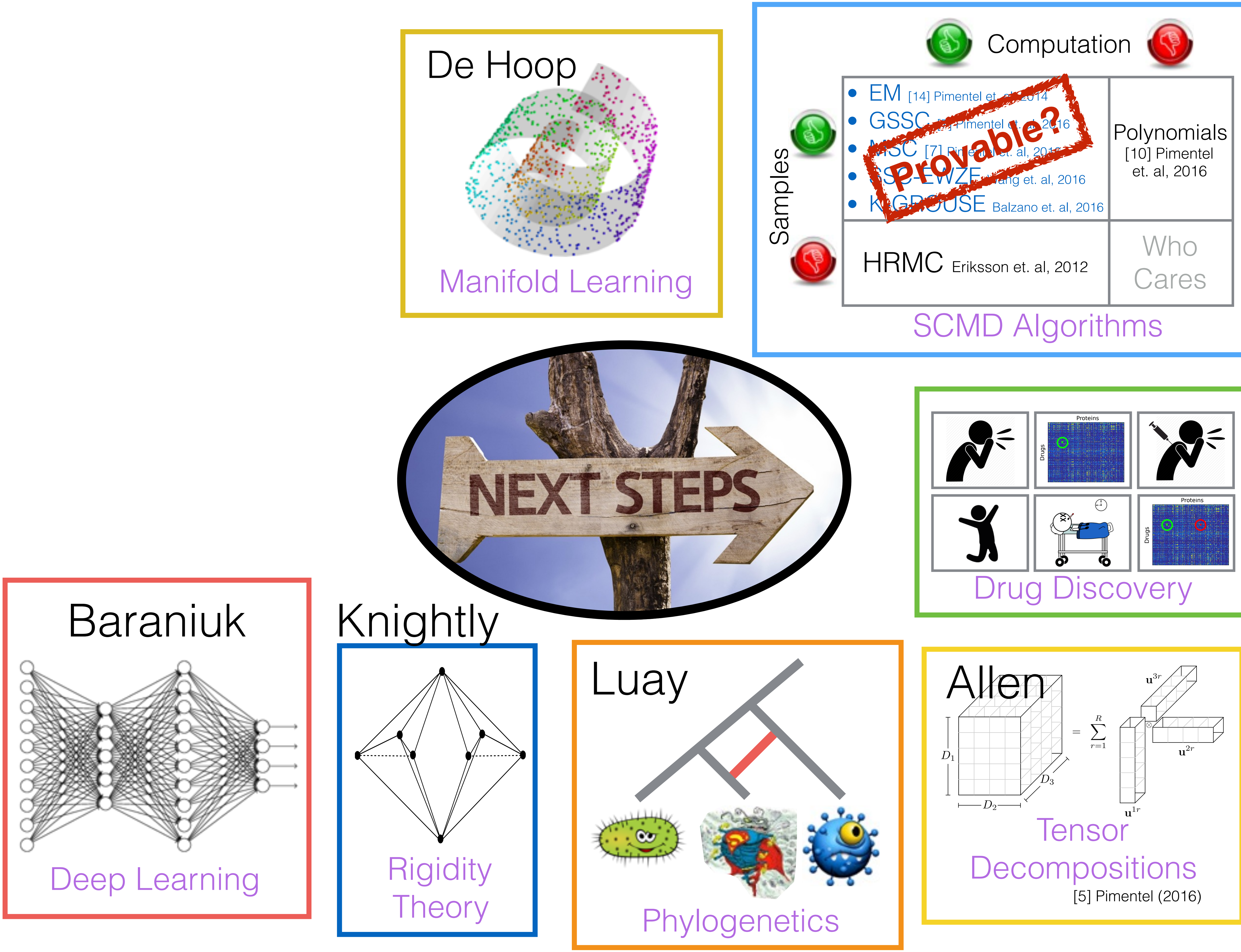
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Grandidier's Baobab
Quiver Tree
Monkey Puzzle Tree
Wood Classification
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Tensor Decompositions
[5] Pimentel (2016)


Nigel Boston


Rob Nowak


Steve Wright

Joint work with:


Becca Willett


Laura Balzano


Roummel Marcia


Claudia Solís


Ari Biswas

Thank you

1. D. Pimentel-Alarcón, A. Biswas and C. Solís-Lemus, *Adversarial Principal Component Analysis, submitted, 2017*.
2. D. Pimentel-Alarcón, L. Balzano, R. Marcia, R. Nowak and R. Willett, *Mixture Regression as Subspace Clustering, submitted, 2017*.
3. D. Pimentel-Alarcón and R. Nowak, *Random Consensus Robust PCA, AISTATS, 2017*.
4. D. Pimentel-Alarcón, L. Balzano and R. Nowak, *Necessary and Sufficient Conditions for Sketched Subspace Clustering*, Allerton, 2016.
5. D. Pimentel-Alarcón, *A Simpler Approach to Low-Rank Tensor Canonical Polyadic Decomposition*, Allerton, 2016.
6. D. Pimentel-Alarcón and C. Solís-Lemus, *Crime Detection via Crowdsourcing*, Mexican Conference on Pattern Recognition, 2016.
7. D. Pimentel-Alarcón, L. Balzano, R. Marcia, R. Nowak and R. Willett, *Group-Sparse Subspace Clustering with Missing Data*, IEEE Statistical Signal Processing, 2016.
8. D. Pimentel-Alarcón and R. Nowak, *A Converse to Low-Rank Matrix Completion*, IEEE International Symposium on Information Theory, 2016.
9. D. Pimentel-Alarcón and R. Nowak, *The Information-Theoretic Requirements of Subspace Clustering with Missing Data*, International Conference on Machine Learning, 2016.
10. D. Pimentel-Alarcón, N. Boston and R. Nowak, *A Characterization of Deterministic Sampling Patterns for Low-Rank Matrix Completion*, IEEE Journal of Selected Topics in Signal Processing, 2016.
11. D. Pimentel-Alarcón and R. Nowak, *Adaptive Strategy for Restricted-Sampling Noisy Low-Rank Matrix Completion*, CAMSAP, 2015.
12. D. Pimentel-Alarcón, N. Boston and R. Nowak, *A Characterization of Deterministic Sampling Patterns for Low-Rank Matrix Completion*, Allerton, 2015.
13. D. Pimentel-Alarcón, N. Boston and R. Nowak, *Deterministic Conditions for Subspace Identifiability from Incomplete Sampling*, IEEE International Symposium on Information Theory, 2015.
14. D. Pimentel-Alarcón, L. Balzano and R. Nowak, *On the Sample Complexity of Subspace Clustering with Missing Data*, IEEE Statistical Signal Processing, 2014.