A Simpler Approach to Low-Rank Tensor Canonical Polyadic Decomposition

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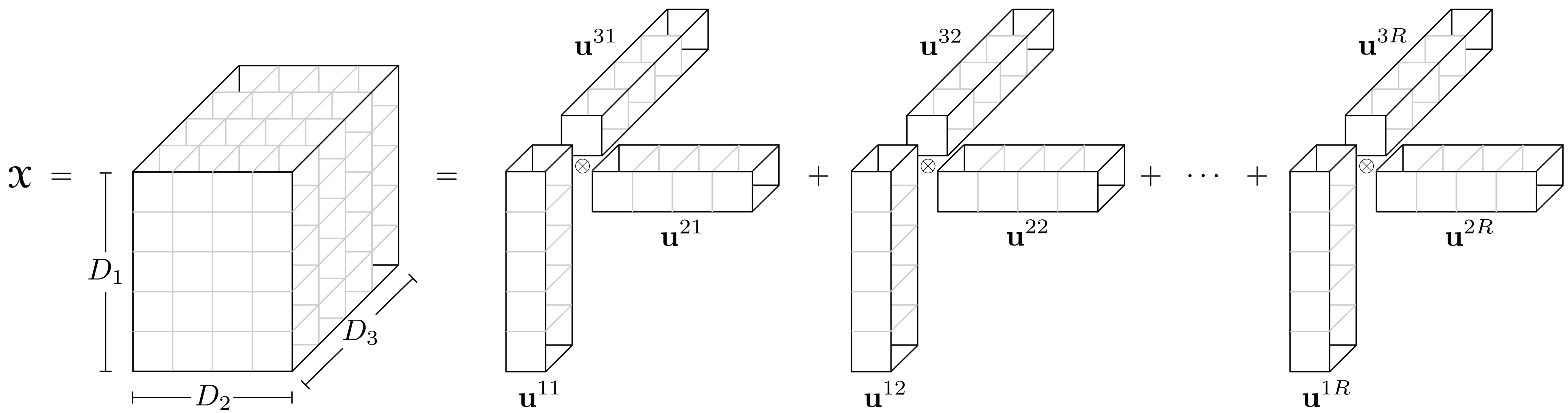
Canonical Polyadic Decomposition

Canonical Polyadic Decomposition

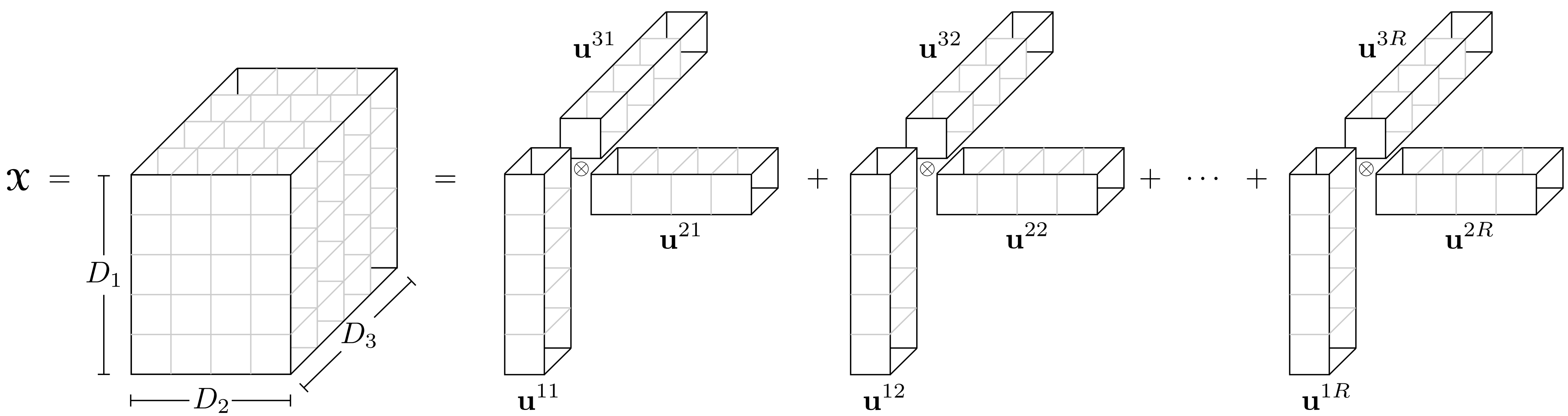
$$\mathcal{X} = \sum_{r=1}^{R} \bigotimes_{k=1}^{K} \mathbf{u}^{kr}$$



Canonical Polyadic Decomposition

Rank

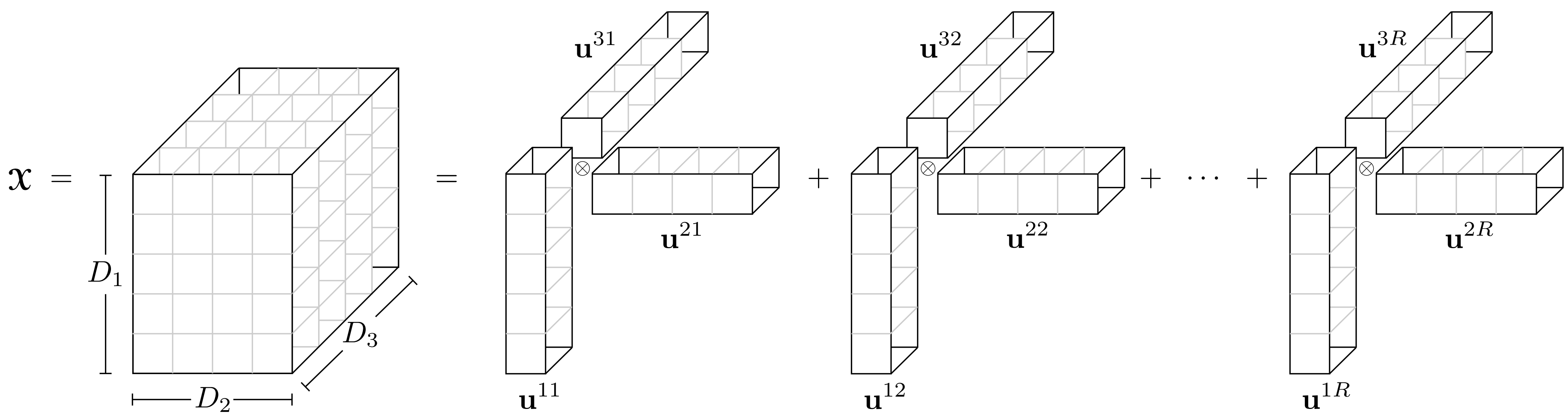
$$\mathcal{X} = \sum_{r=1}^{R} \bigotimes_{k=1}^{K} \mathbf{u}^{kr}$$



Canonical Polyadic Decomposition

Rank Order

$$\mathcal{X} = \sum_{r=1}^{R} \bigotimes_{k=1}^{K} \mathbf{u}^{kr}$$



Canonical Polyadic Decomposition

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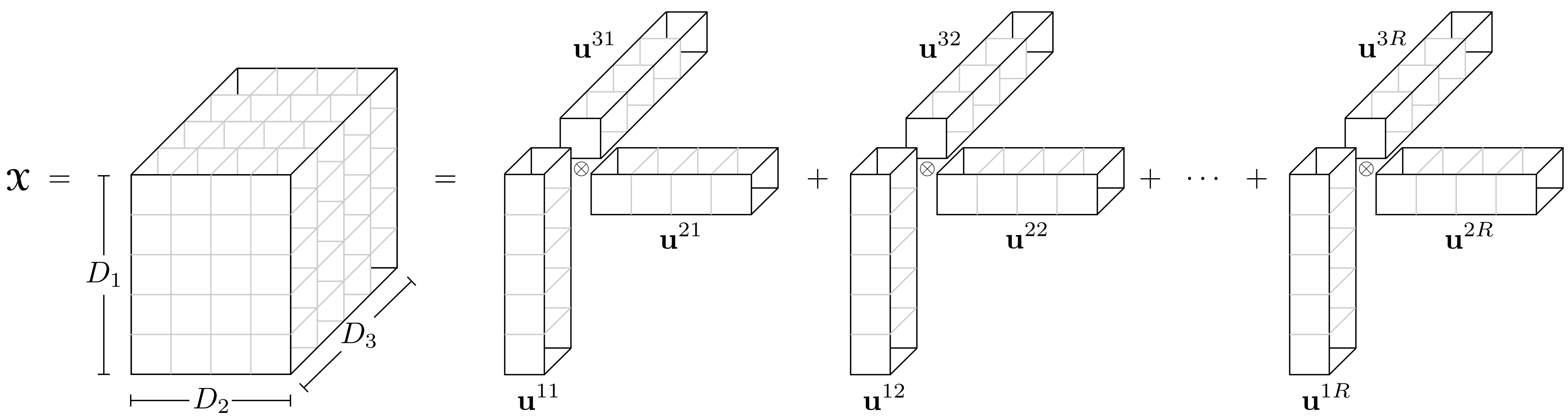

Canonical Polyadic Decomposition

Low-Rank: $R \leq D_2$

Existing Methods

- Alternating Minimization
- Simultaneous Diagonalization
- Generalized Eigenvalue Decomposition
- Line Search

This Talk

Simple Approach - Elemental Linear Algebra

Key Insight

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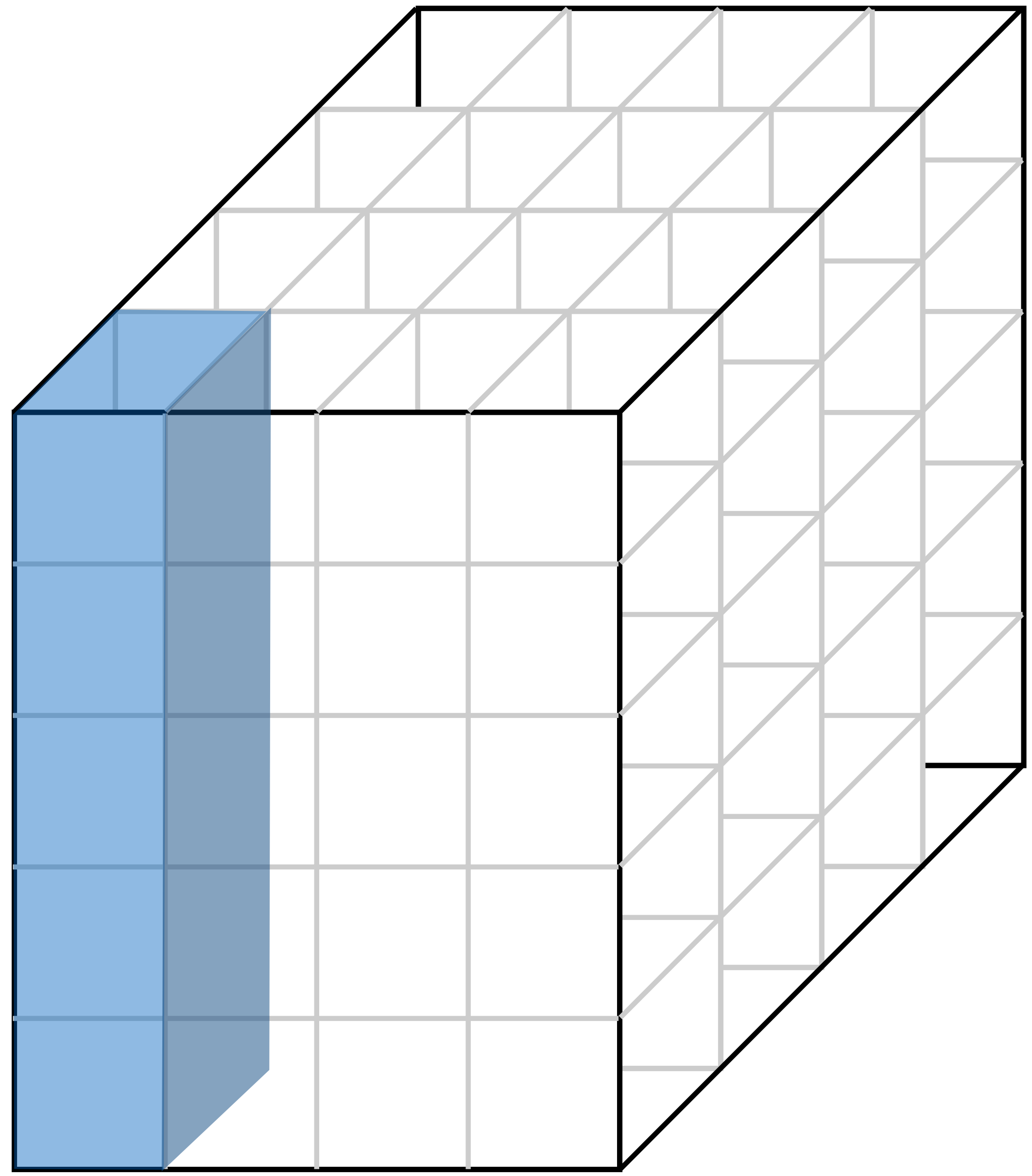
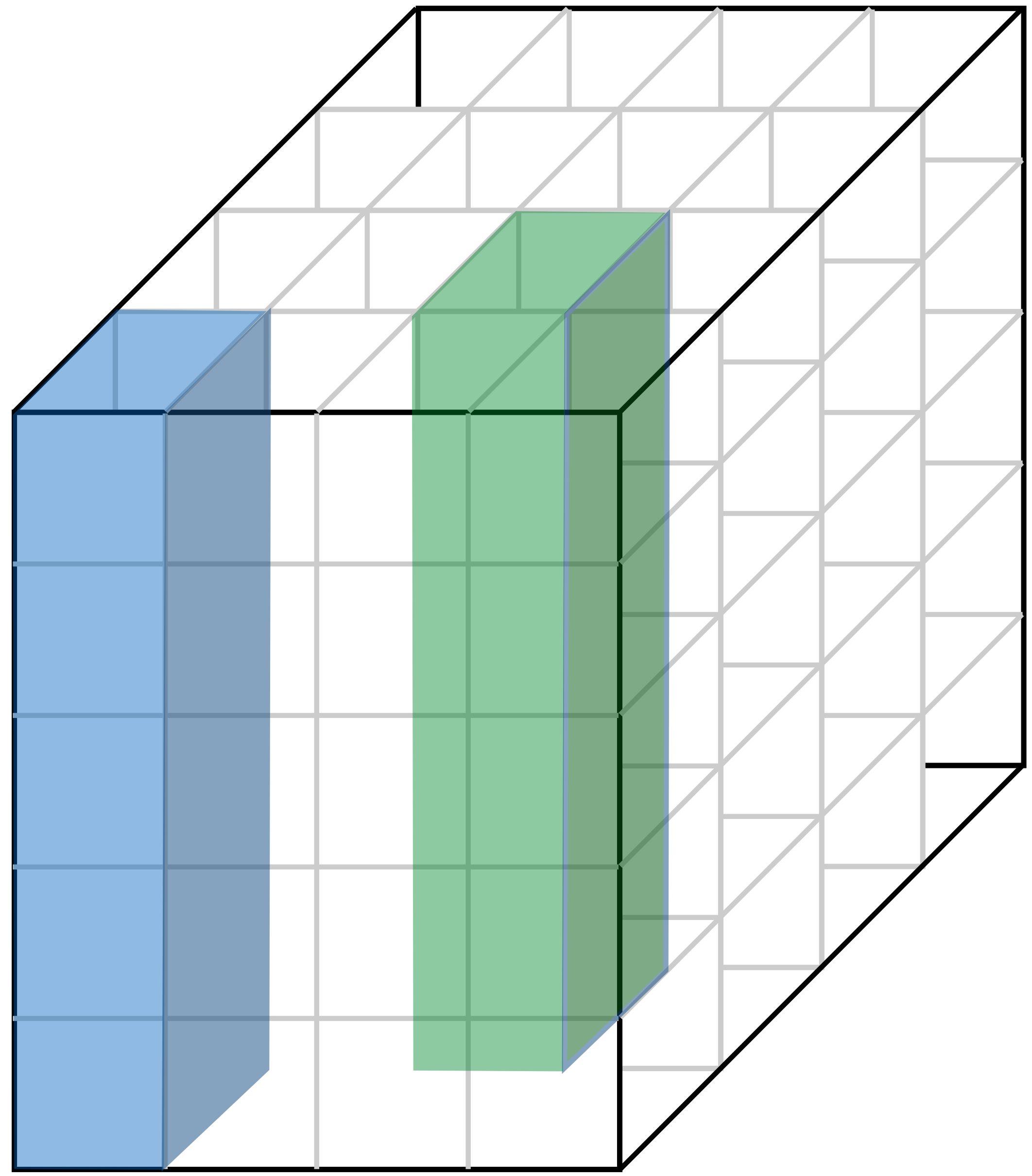


Key Insight

All columns lie in an R-dimensional Subspace



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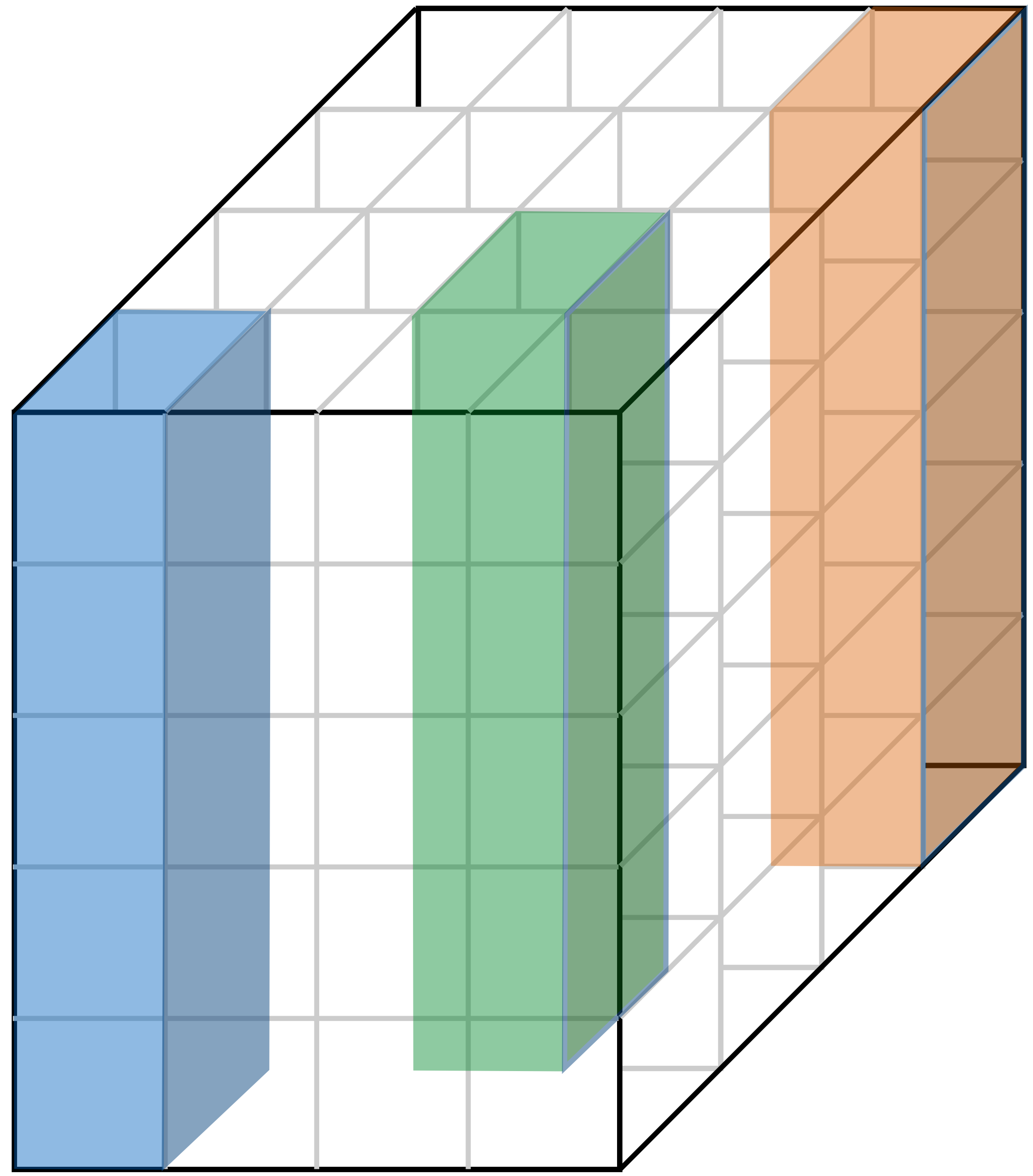

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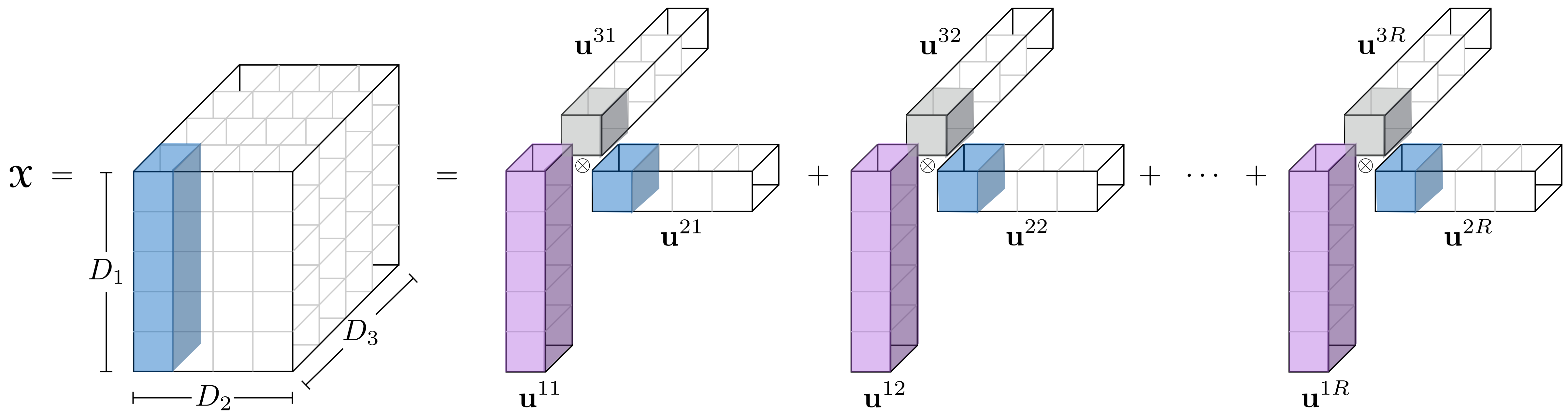

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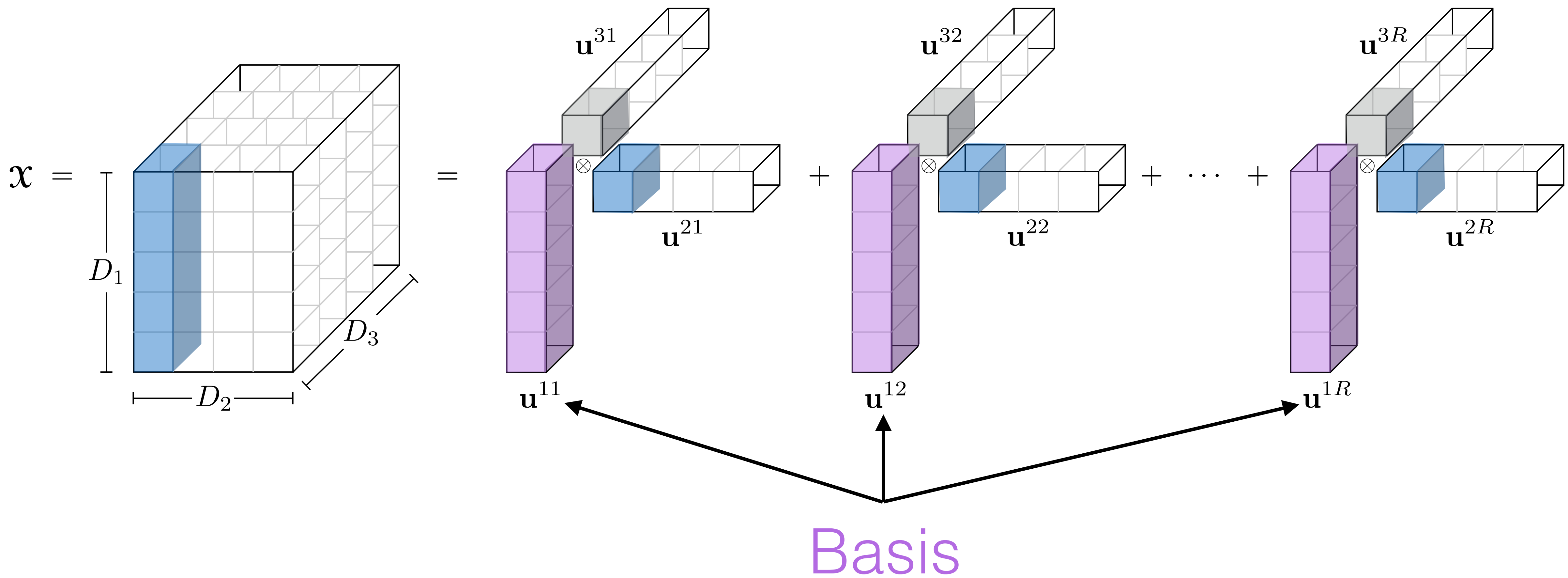

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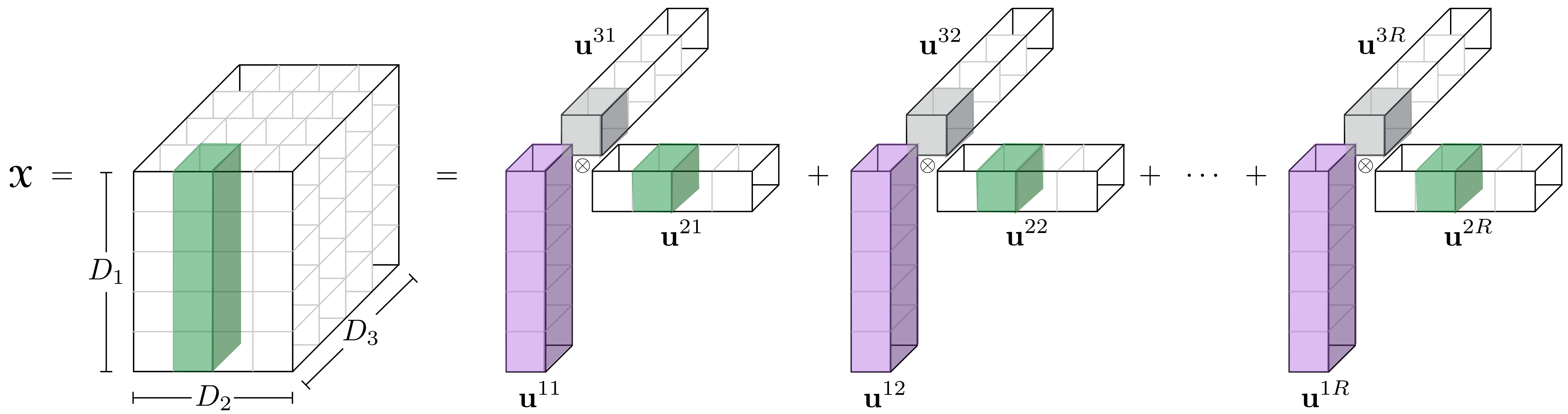

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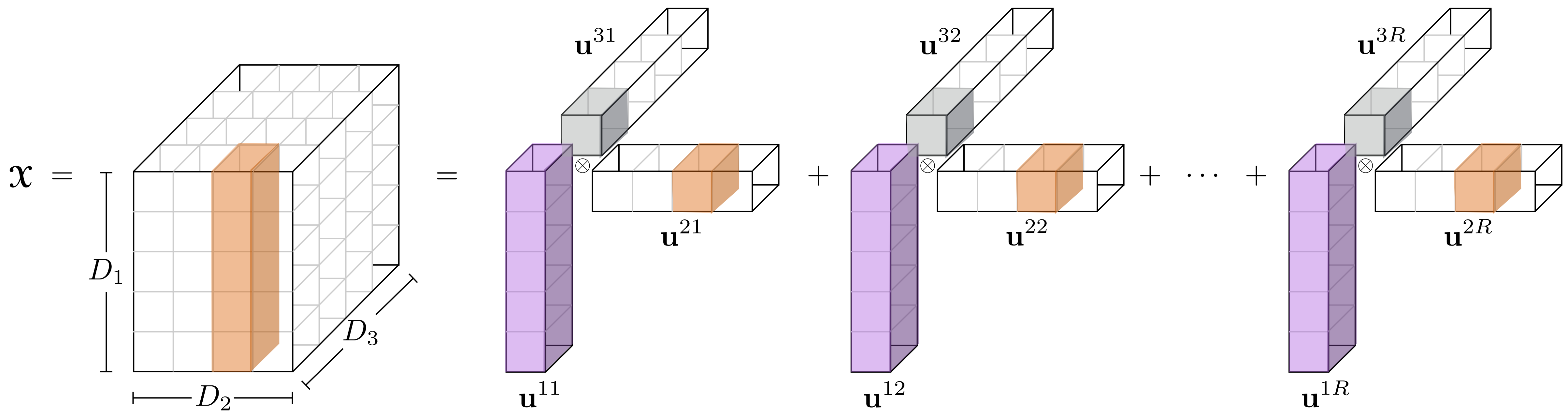

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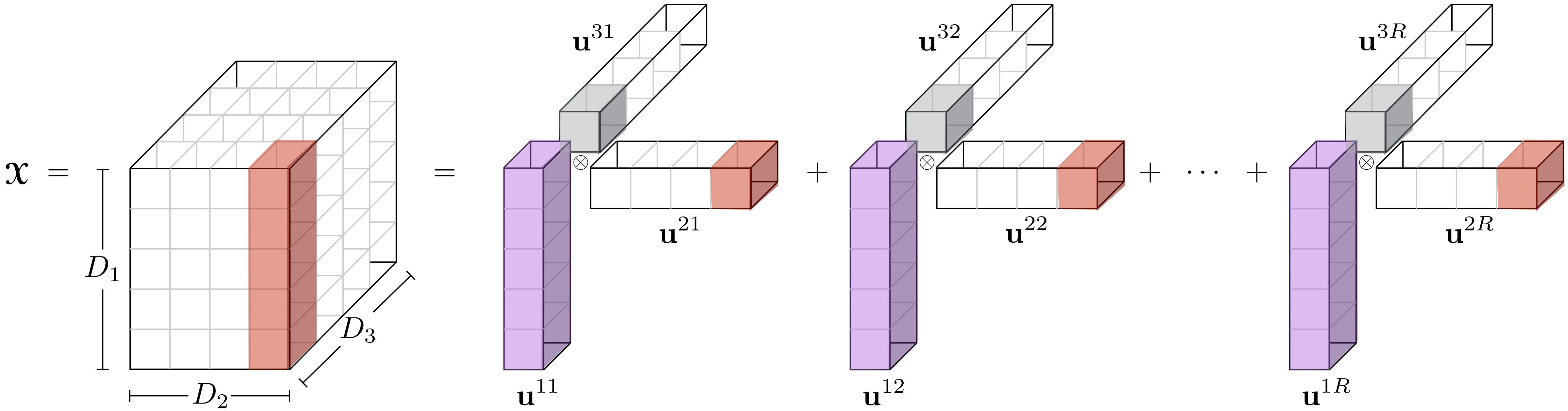

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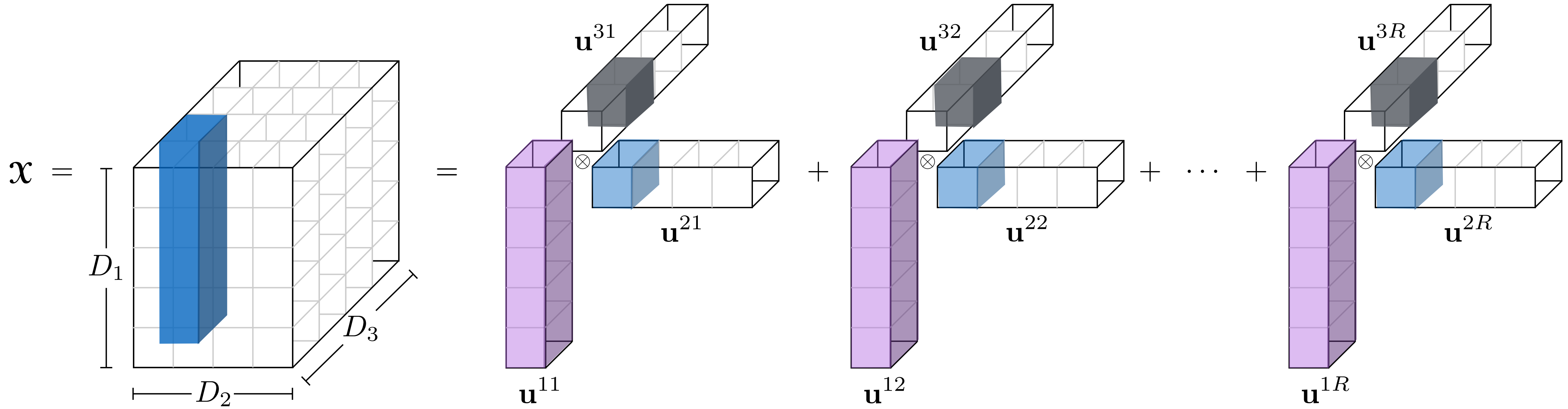

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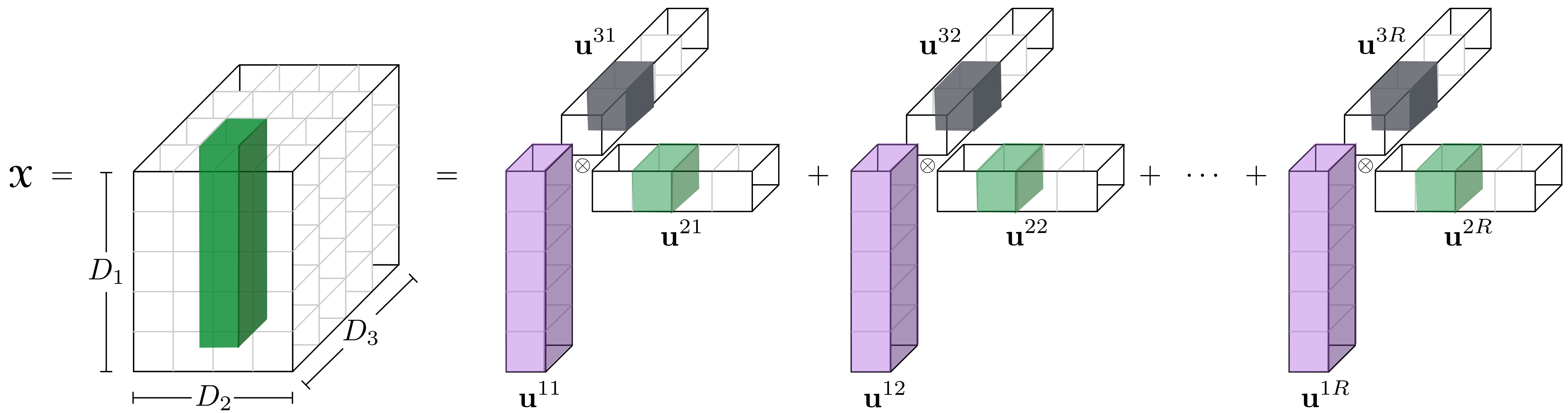

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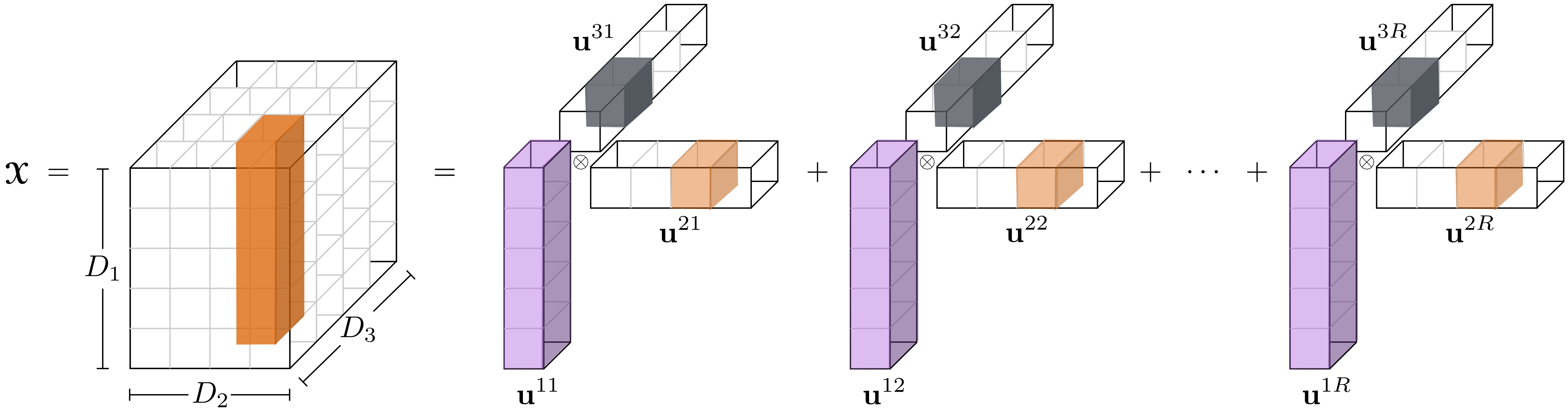

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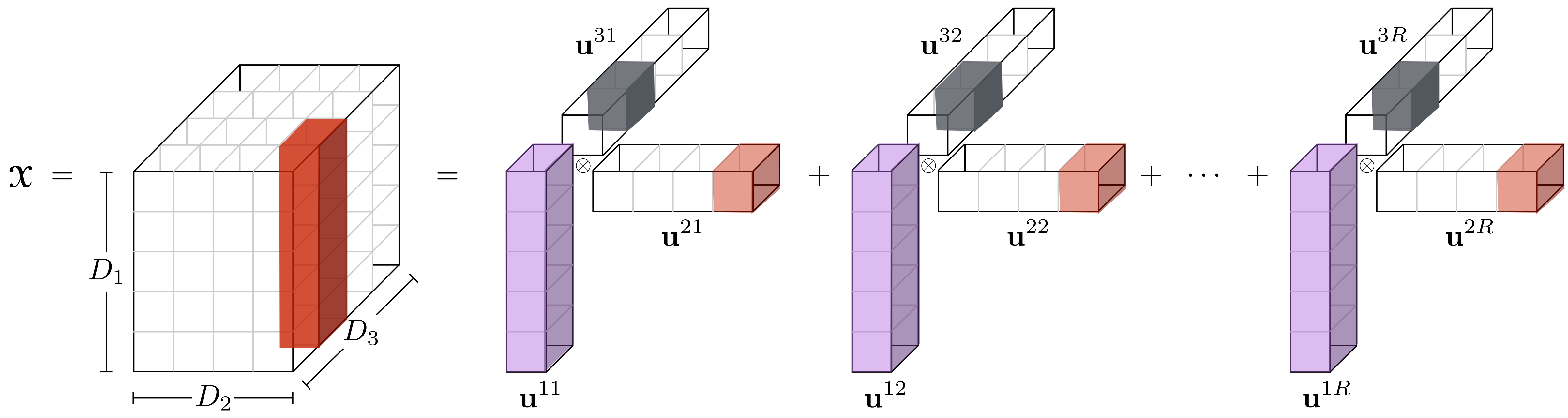

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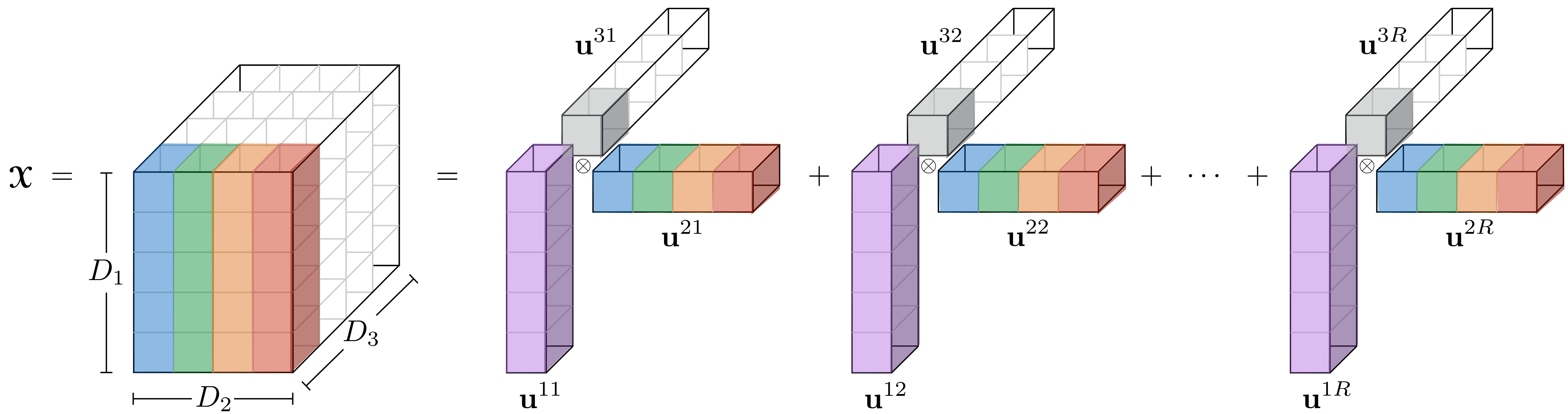

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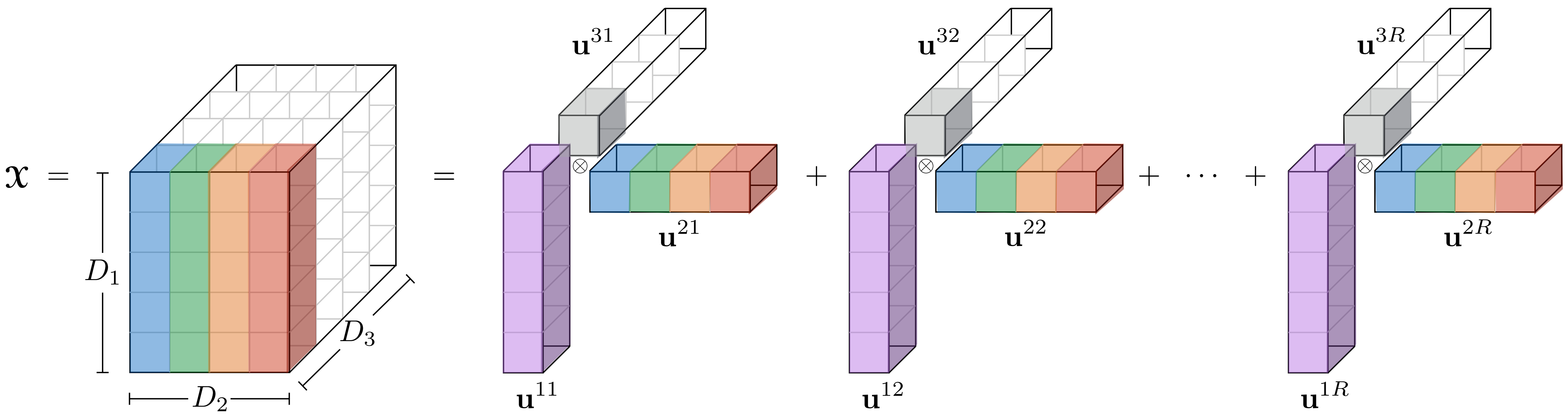

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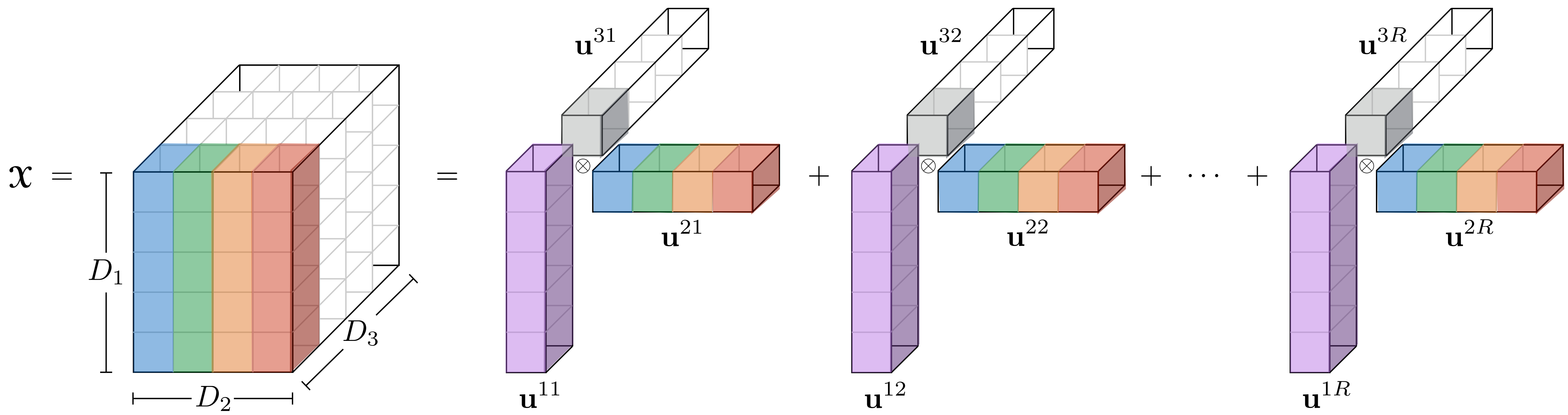

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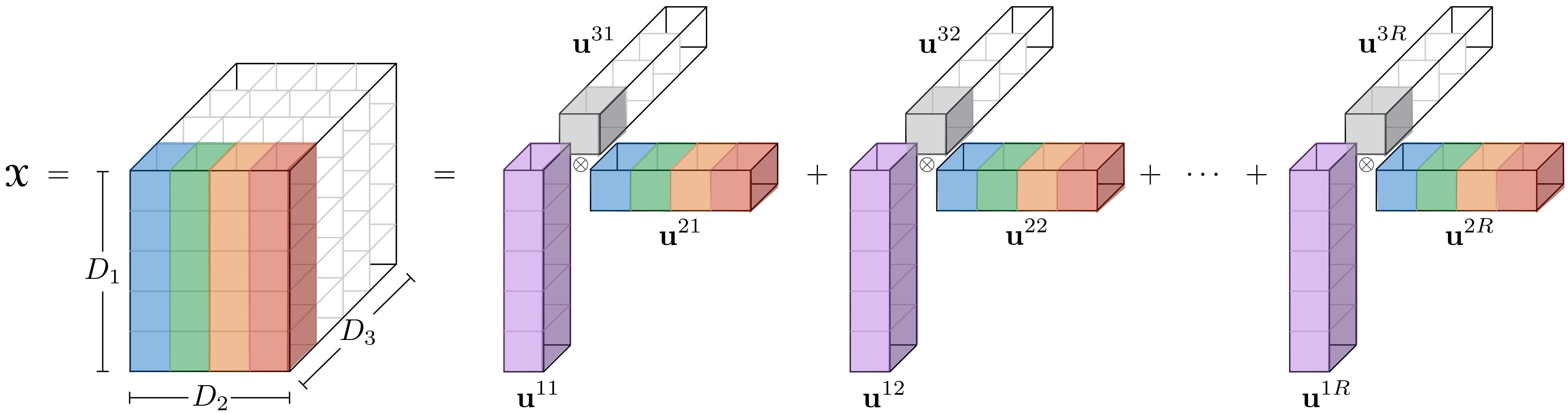

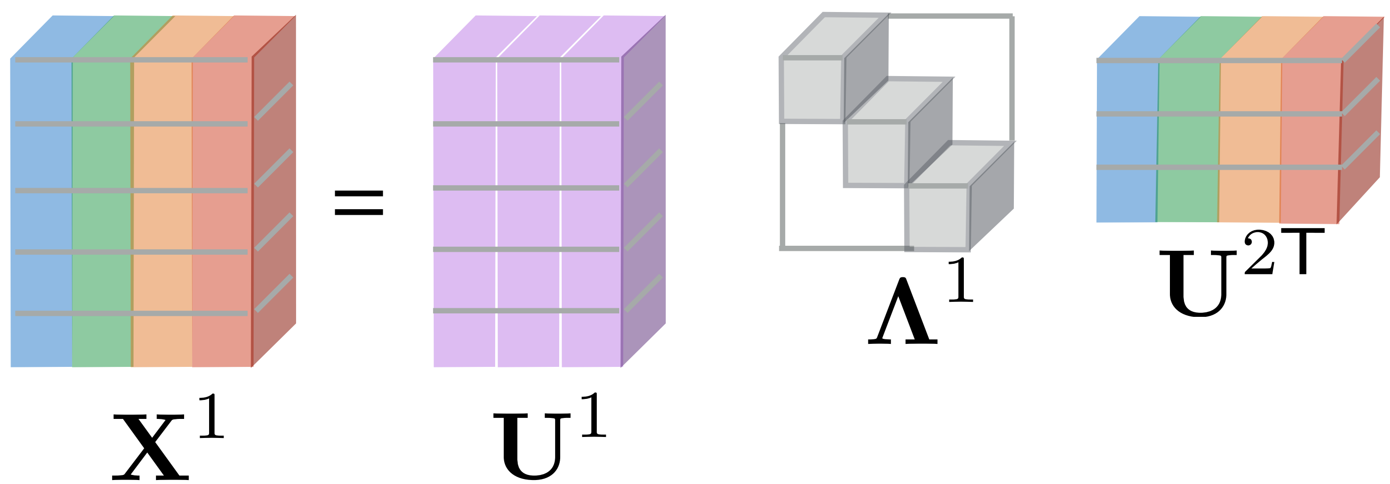

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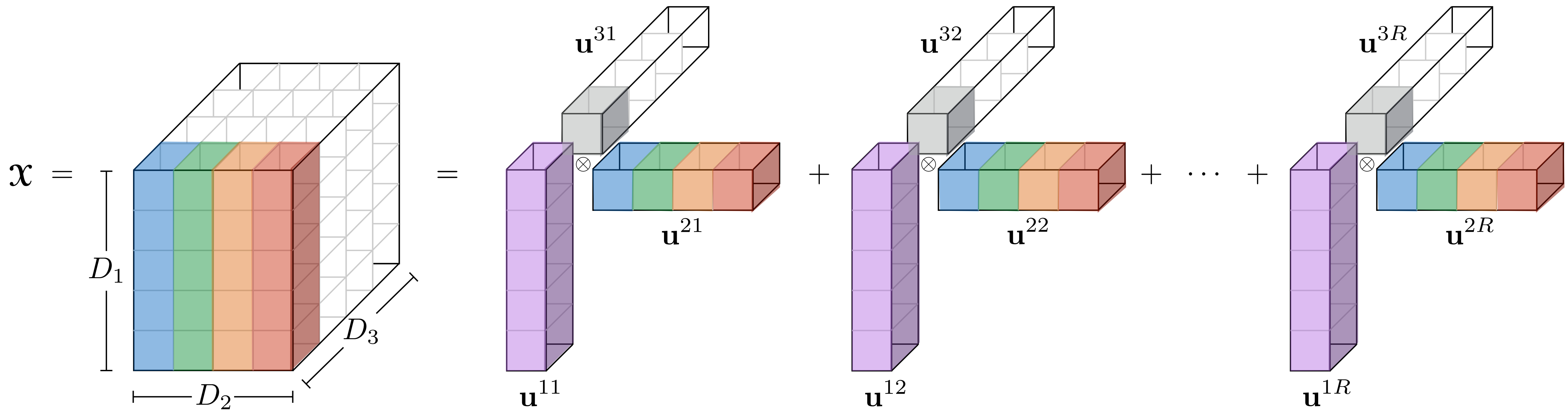

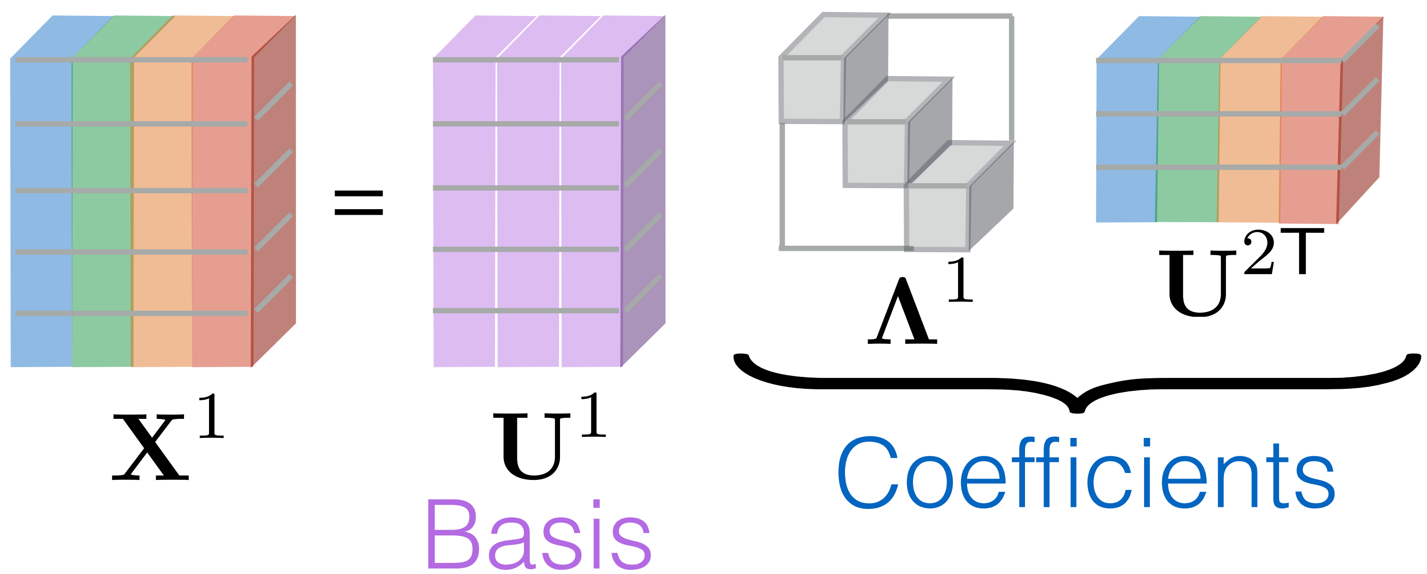

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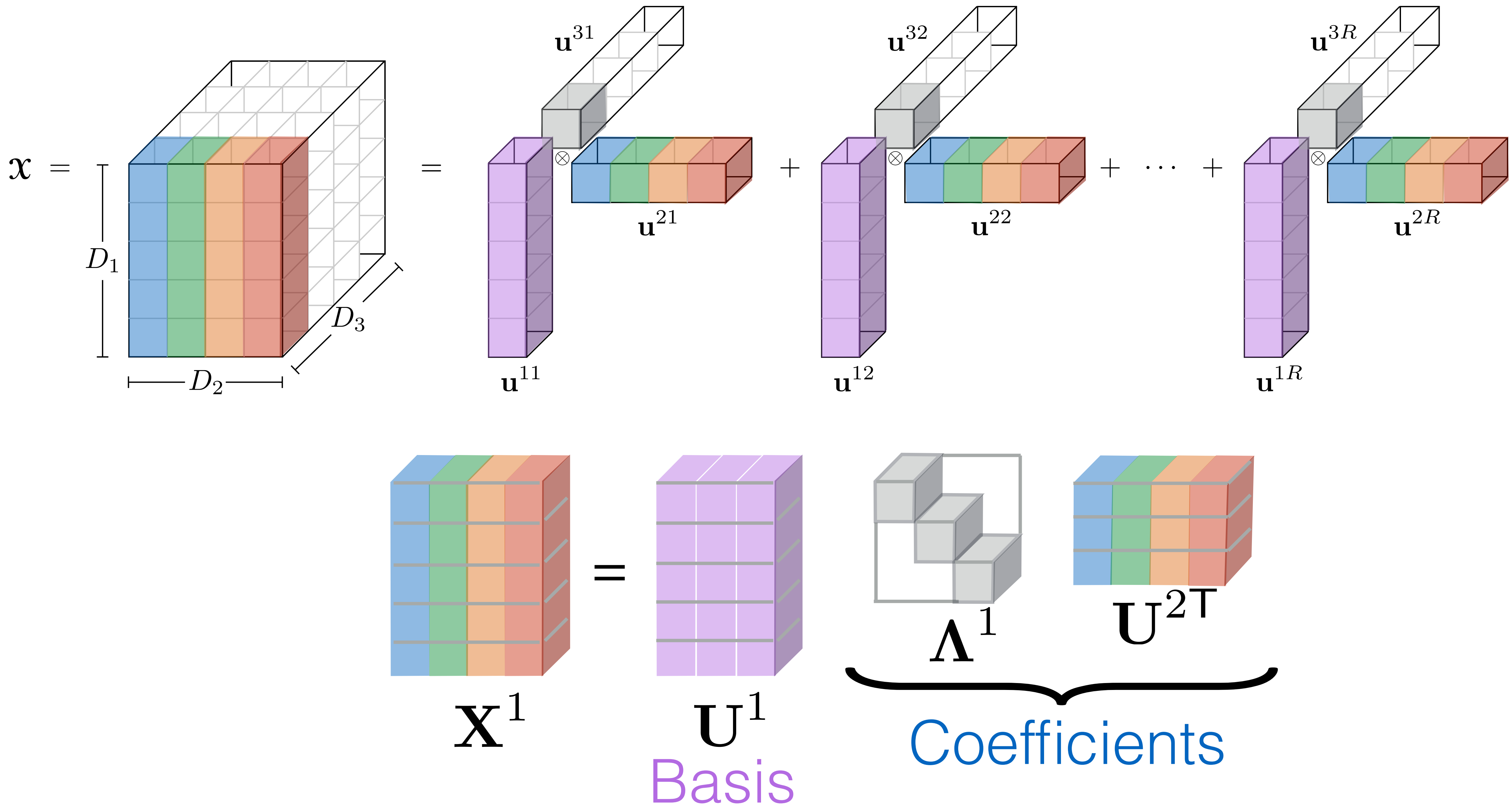
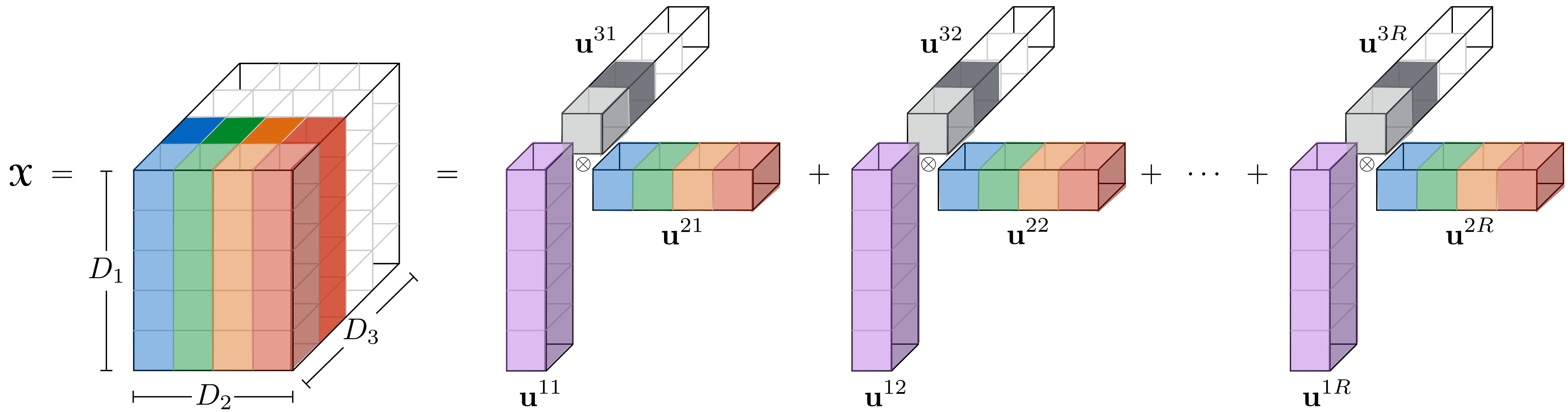
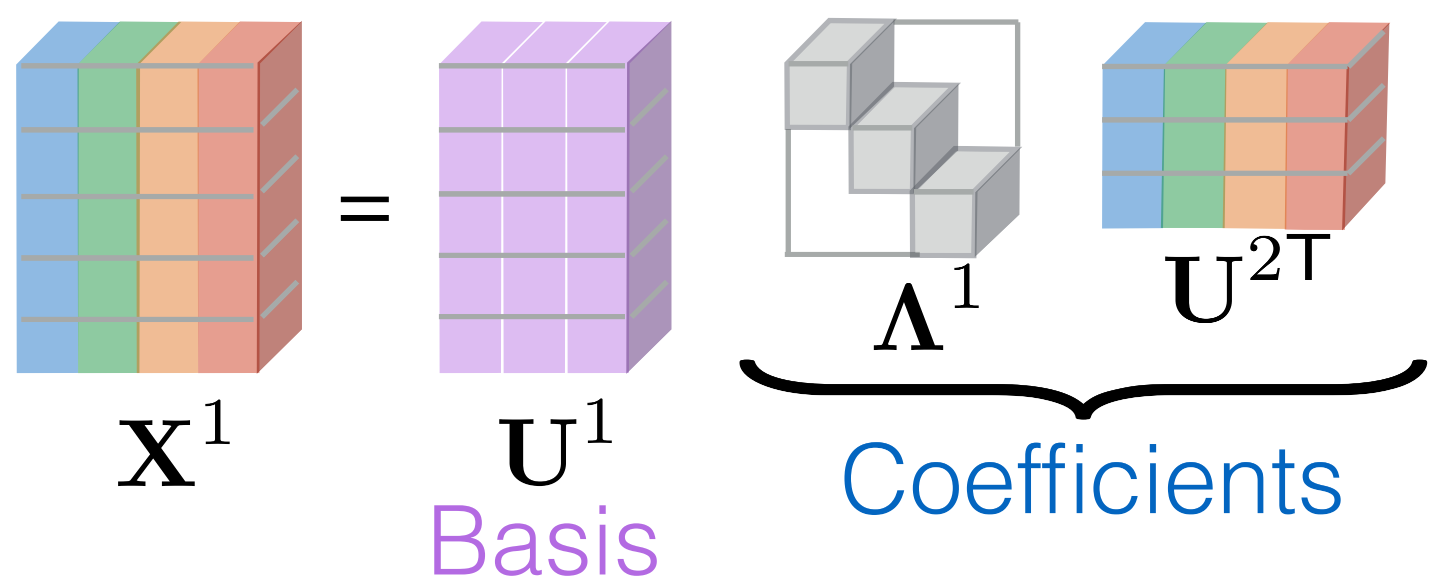


Key Observation

Coefficients of different slices keep a tight relation!





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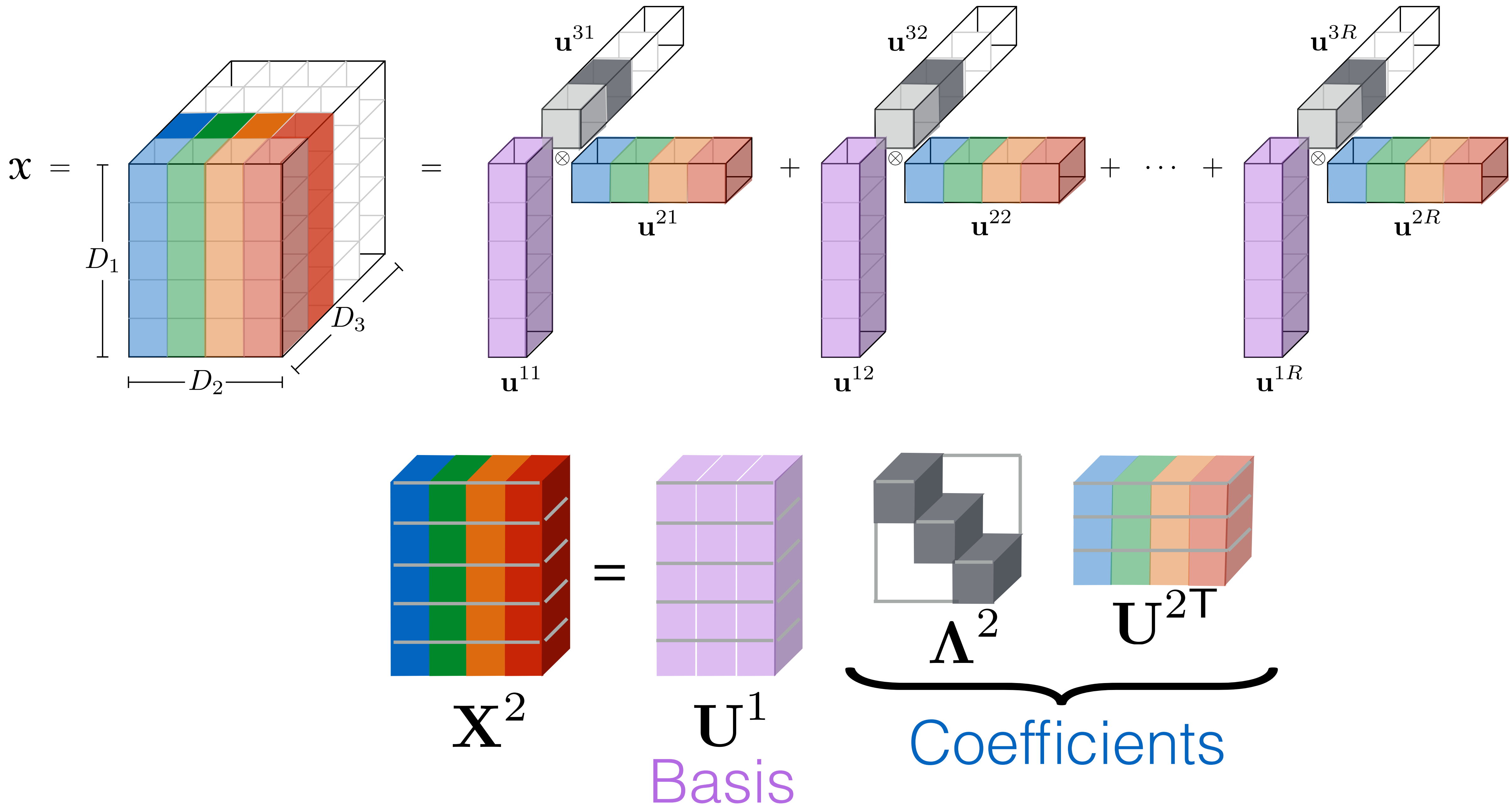

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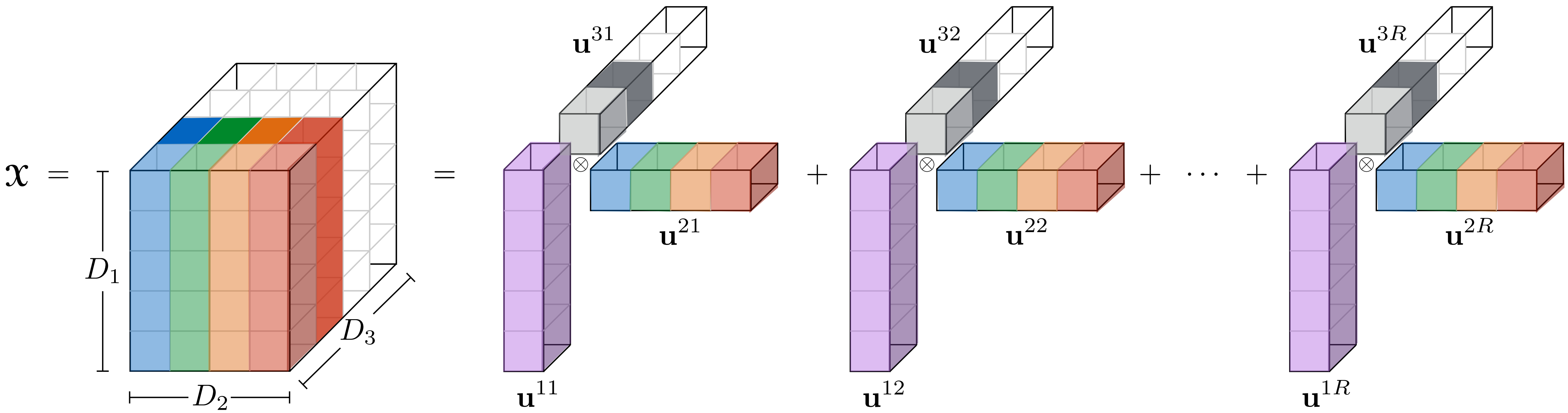
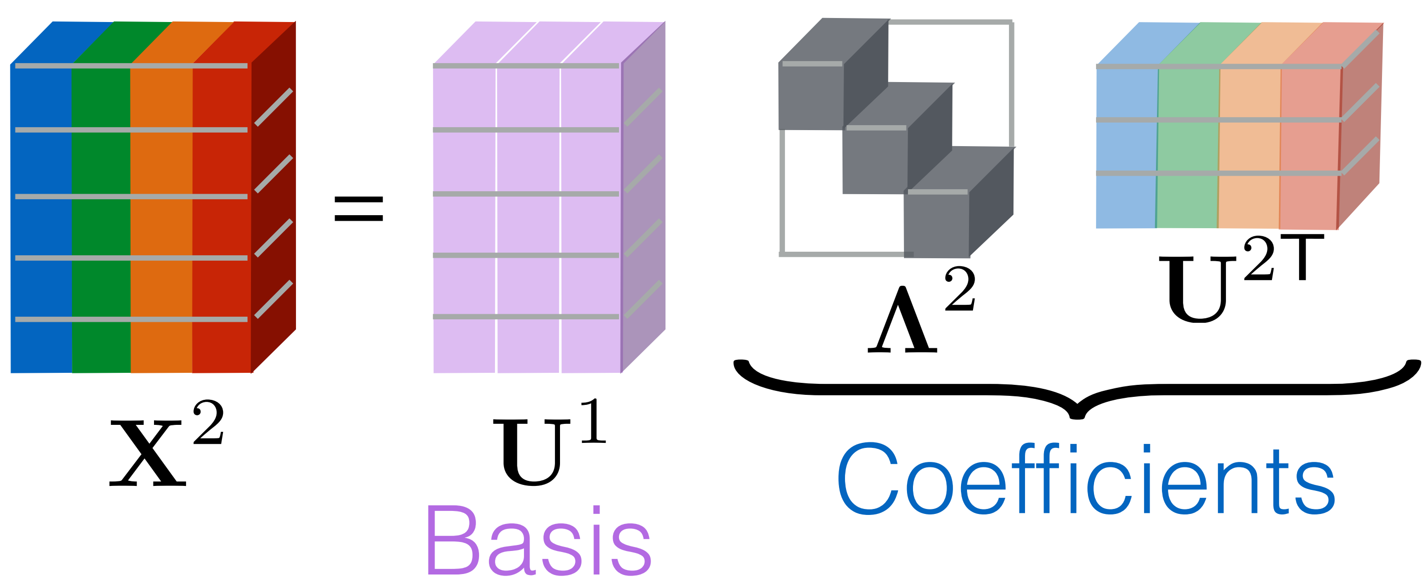


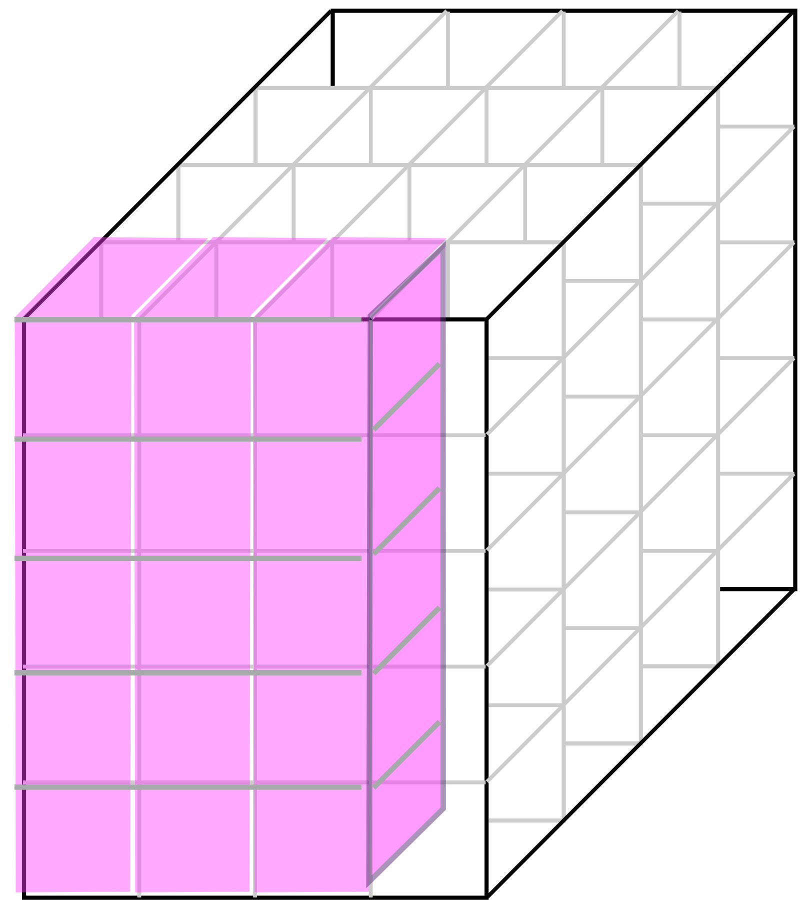


Our Goal

Find Basis and Coefficients

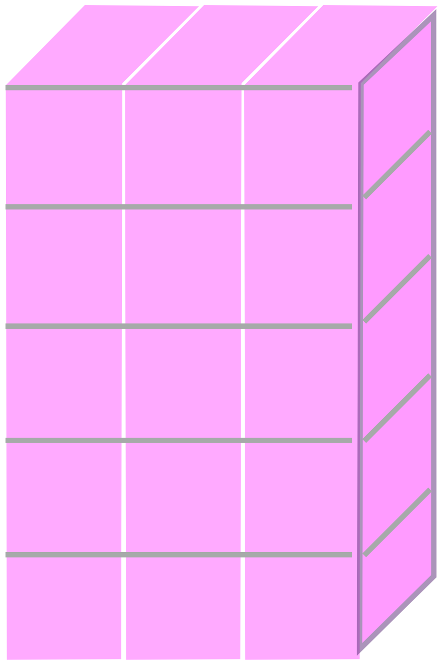
Our Strategy

Our Strategy

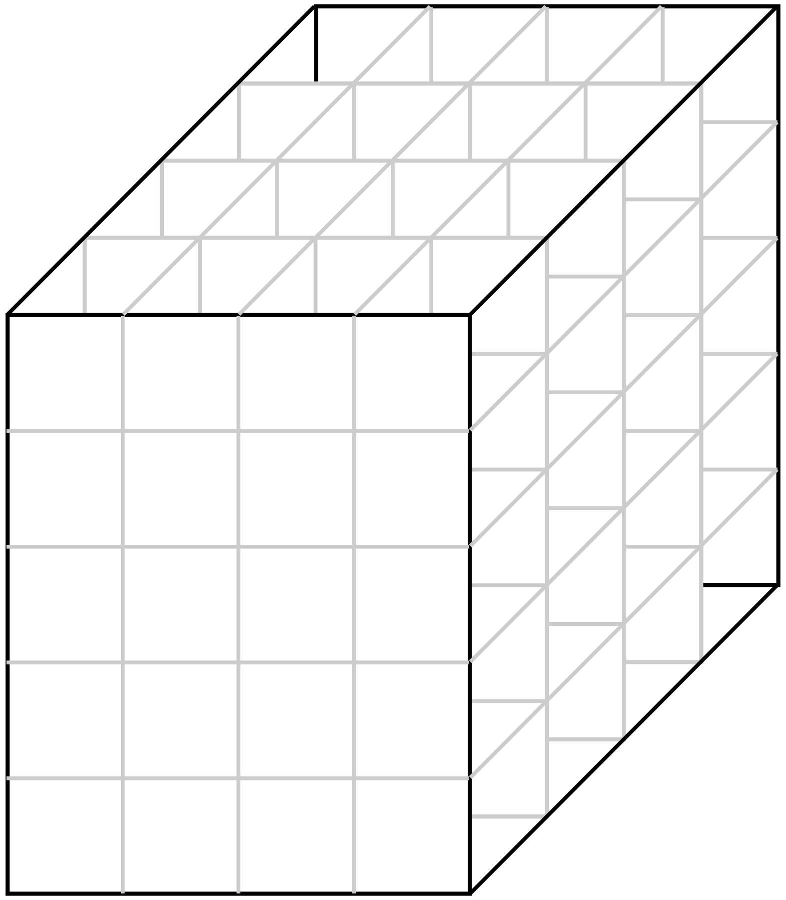


Our Strategy

1. Take *any* basis.
(In this example, R=3)



$\widetilde{\mathbf{U}}^{1}$



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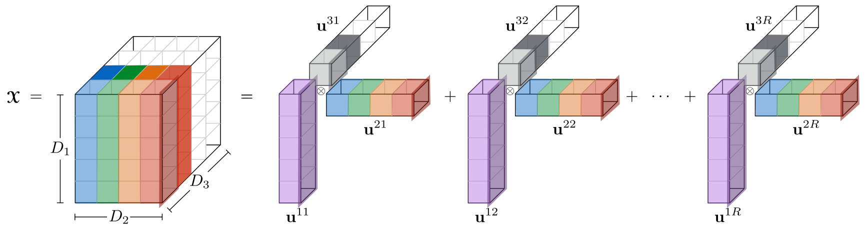

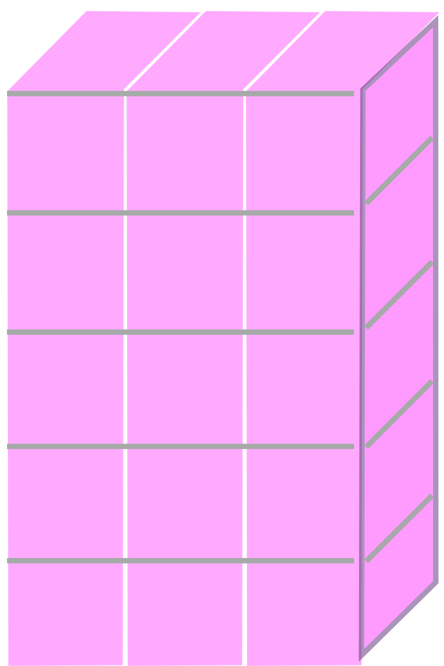

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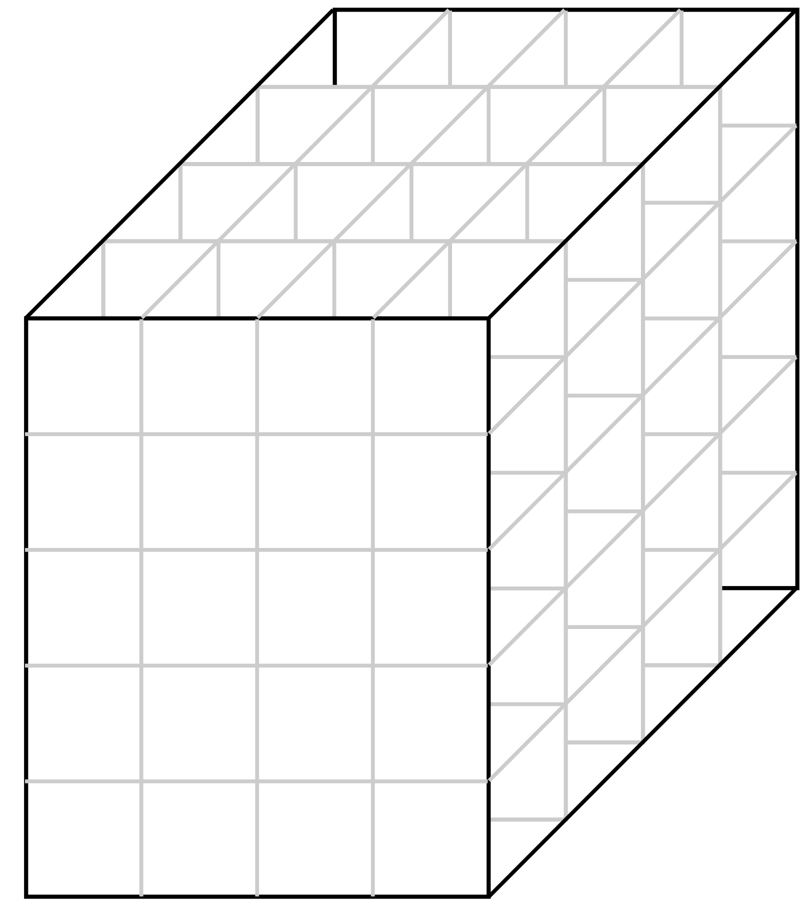

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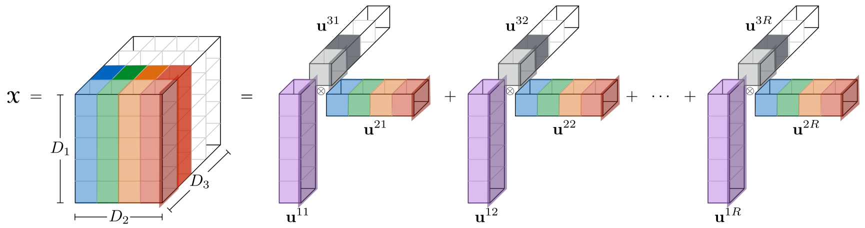
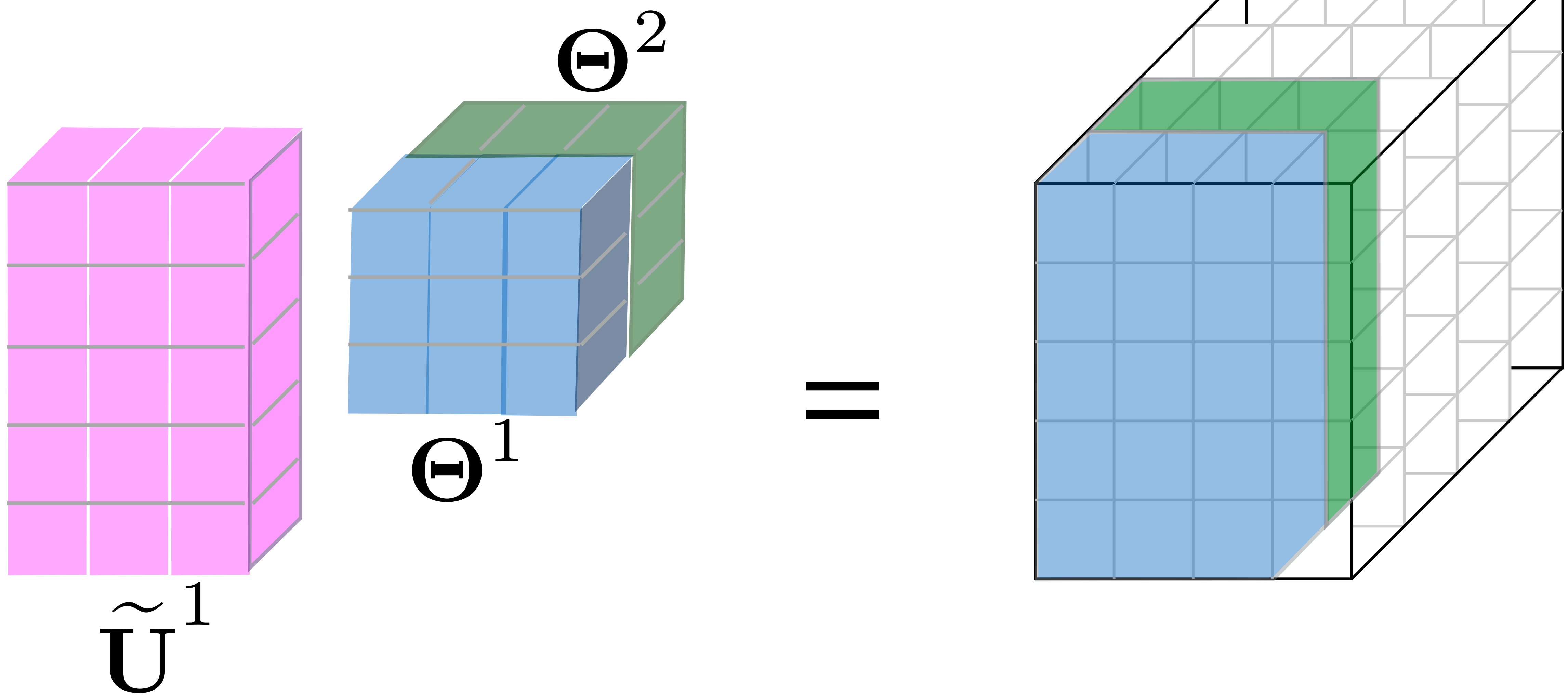
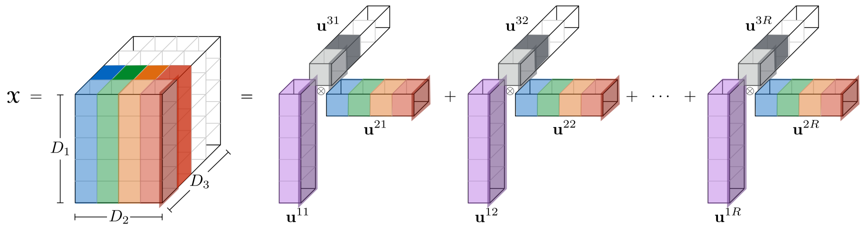
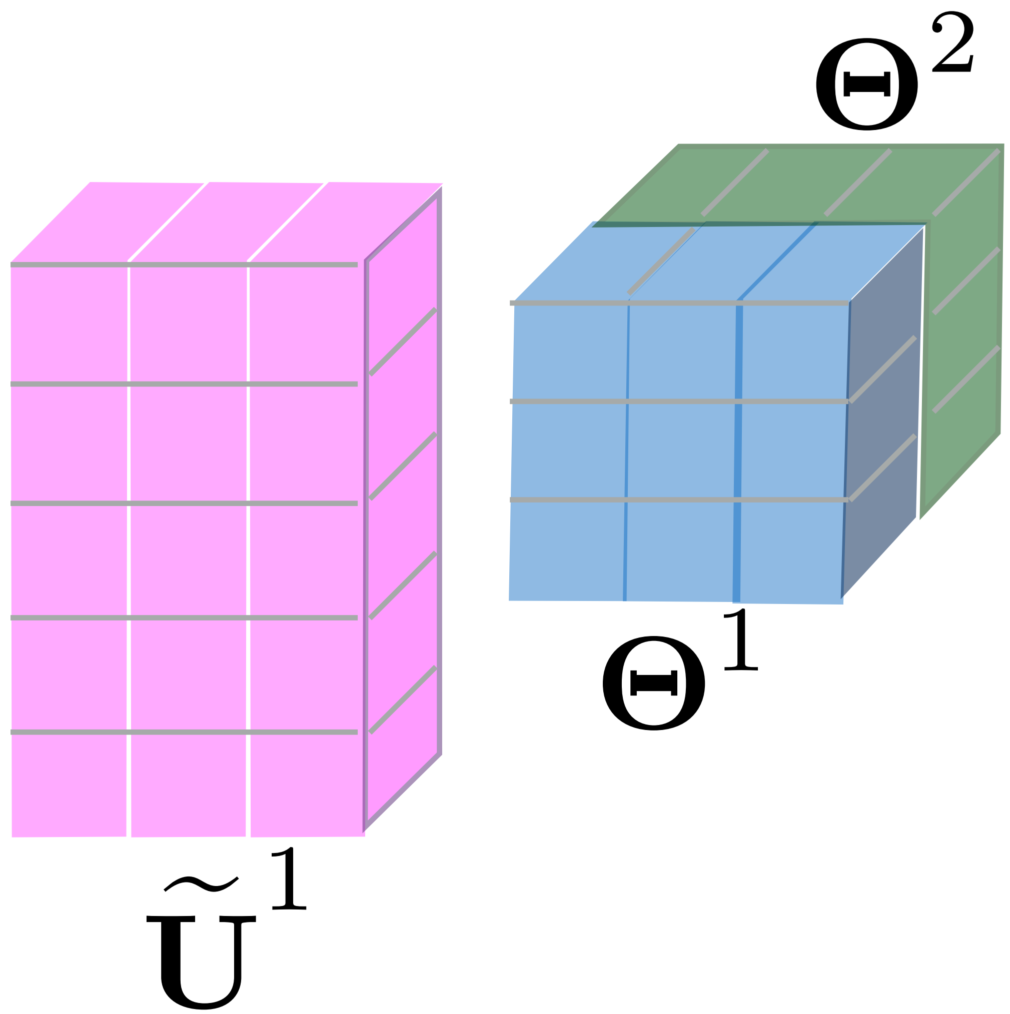




Our Strategy

2. Take coefficients of *R* columns in two slices





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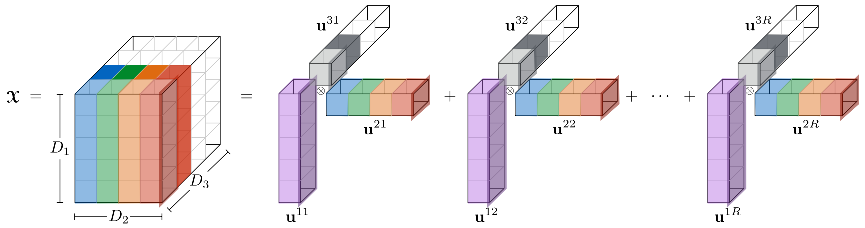
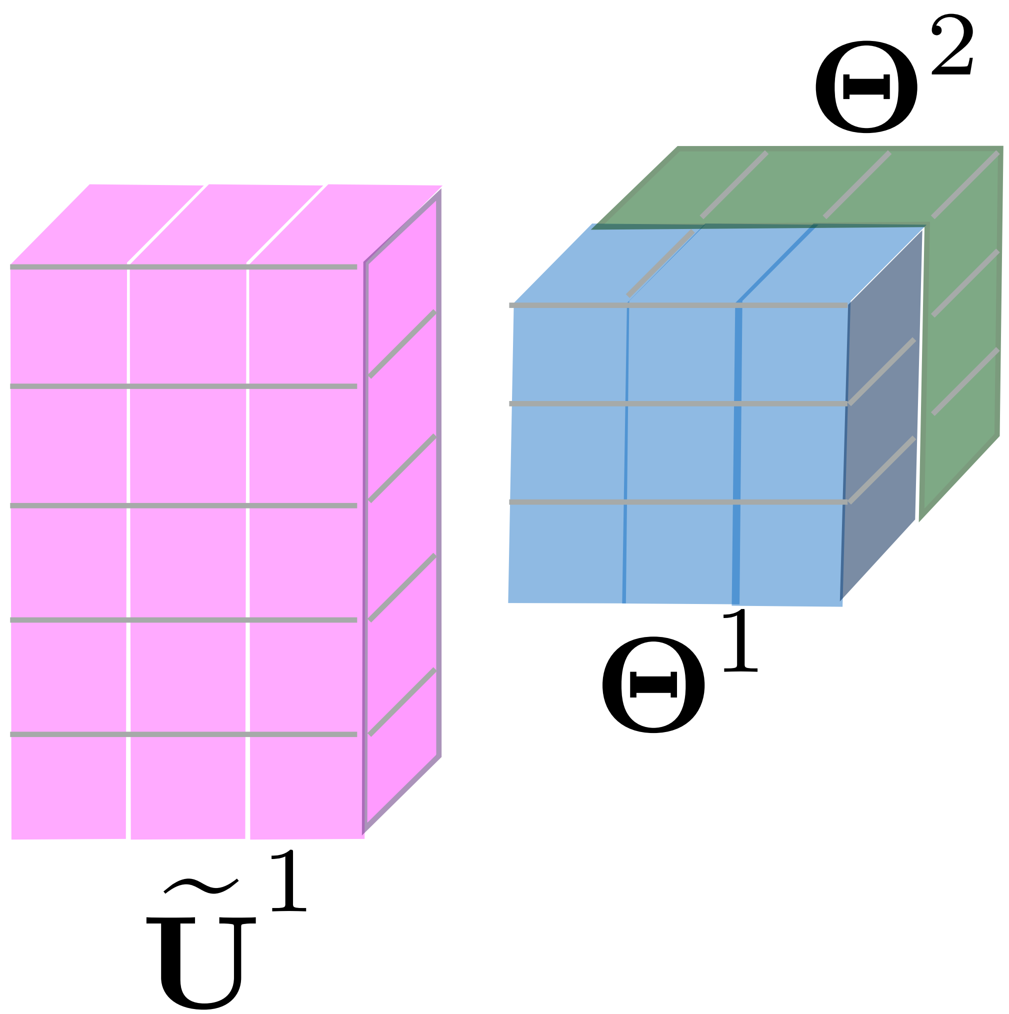
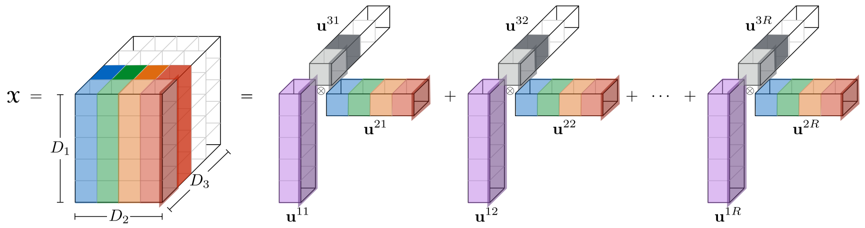
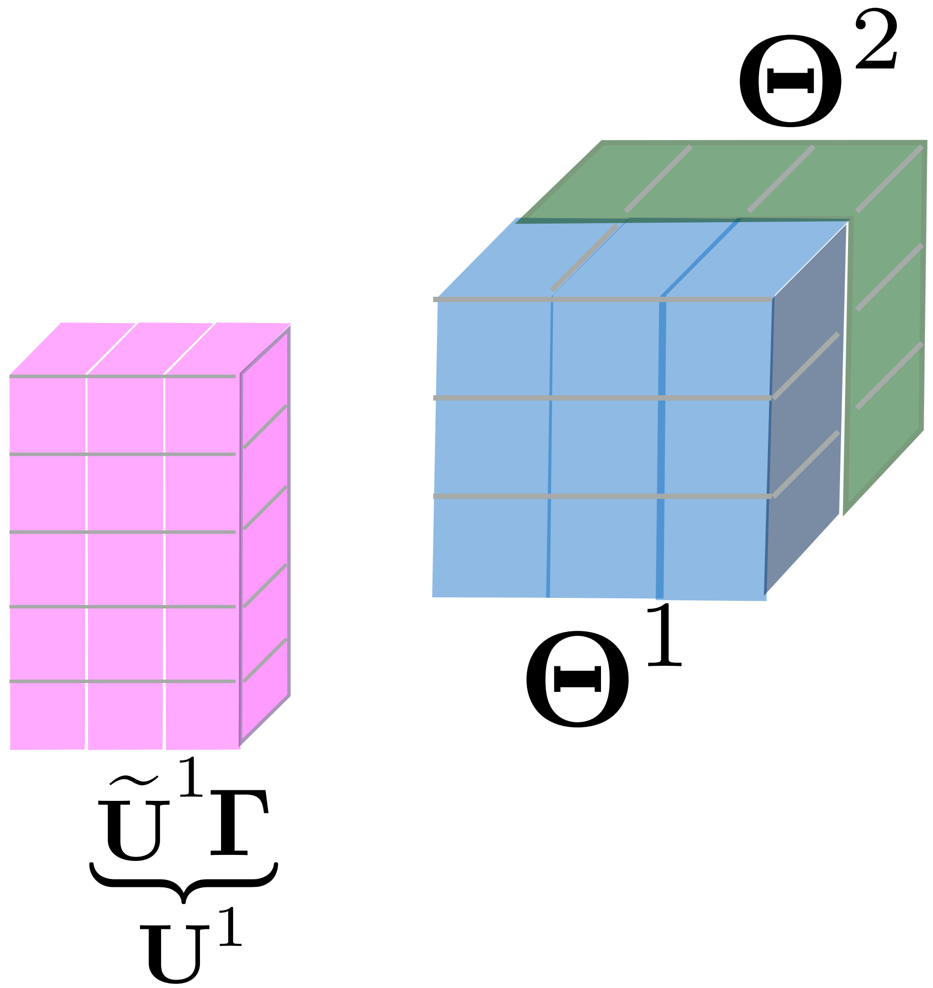




Our Strategy

3. Need to find *the right* change of Basis $\mathbf{\Gamma}$





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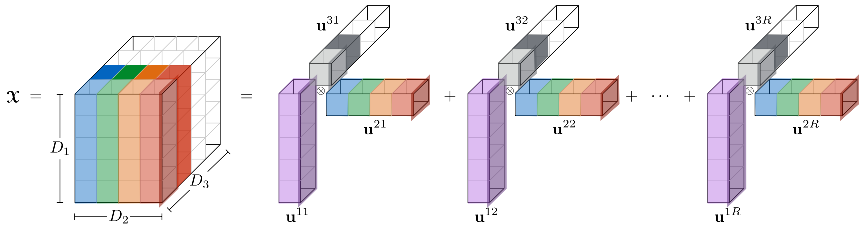

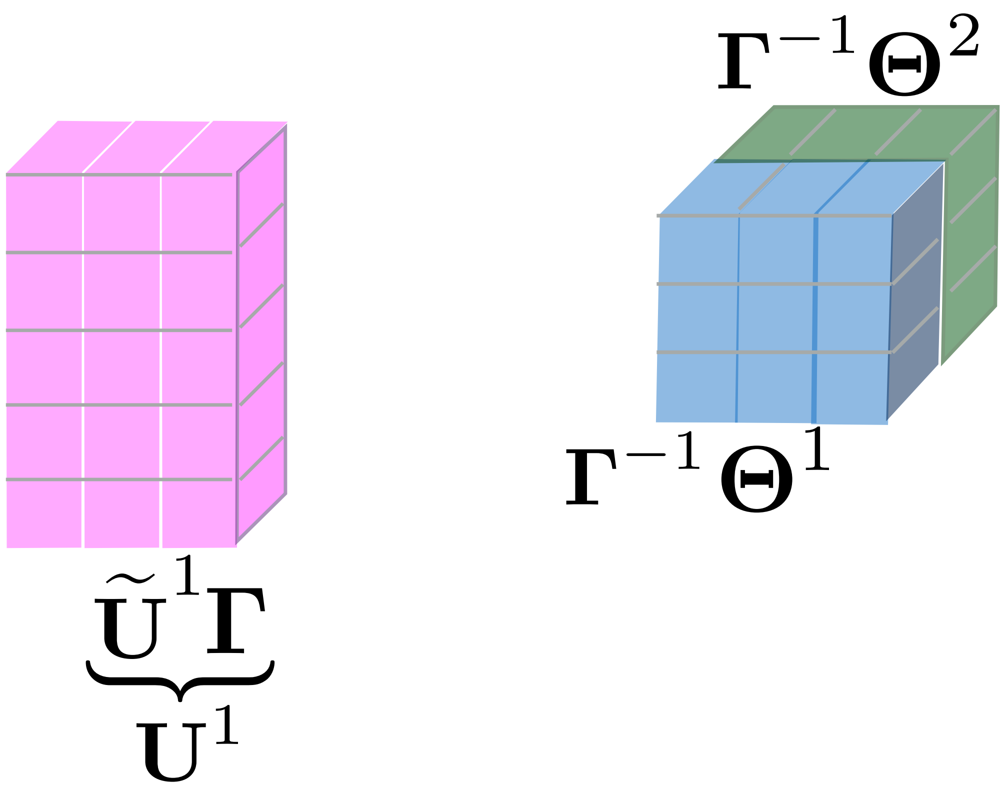

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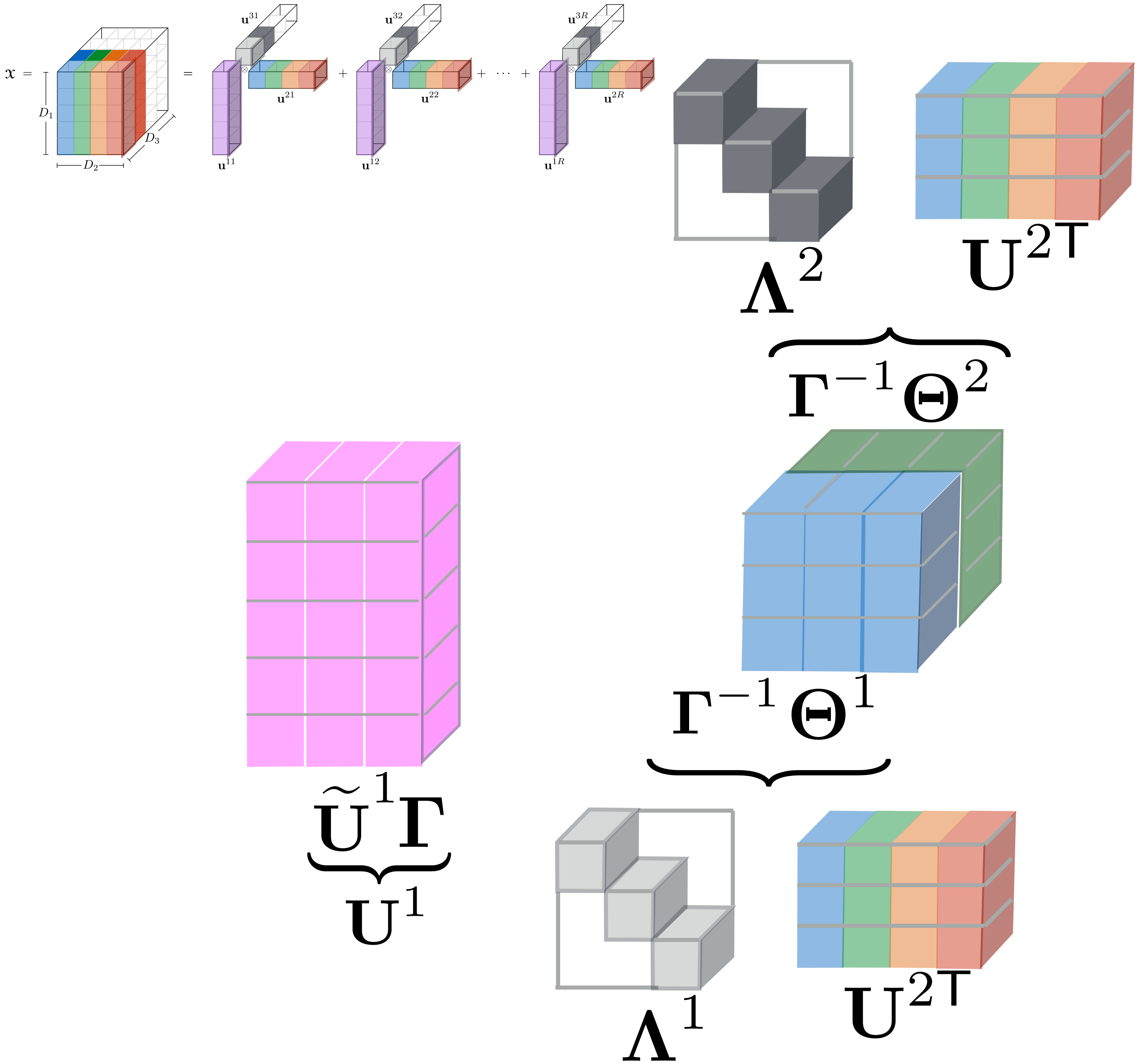

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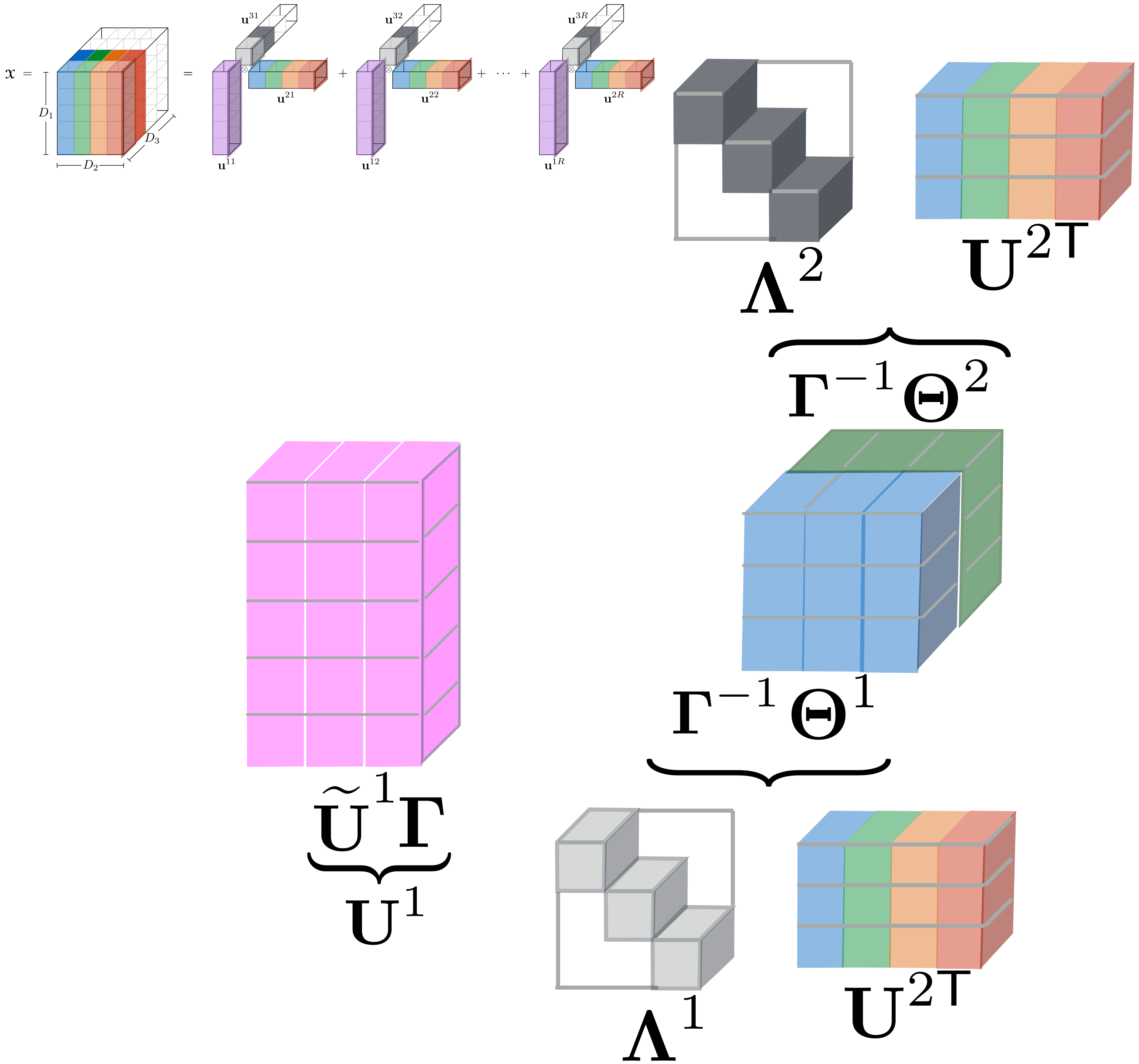

$$\boxed{\begin{aligned} \boldsymbol{\Gamma}^{-1}\boldsymbol{\Theta}^1 &= \boldsymbol{\Lambda}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Theta}^2 \\ \boldsymbol{\Theta}^1(\boldsymbol{\Theta}^2)^{-1} &= \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^{-1} \end{aligned}}$$

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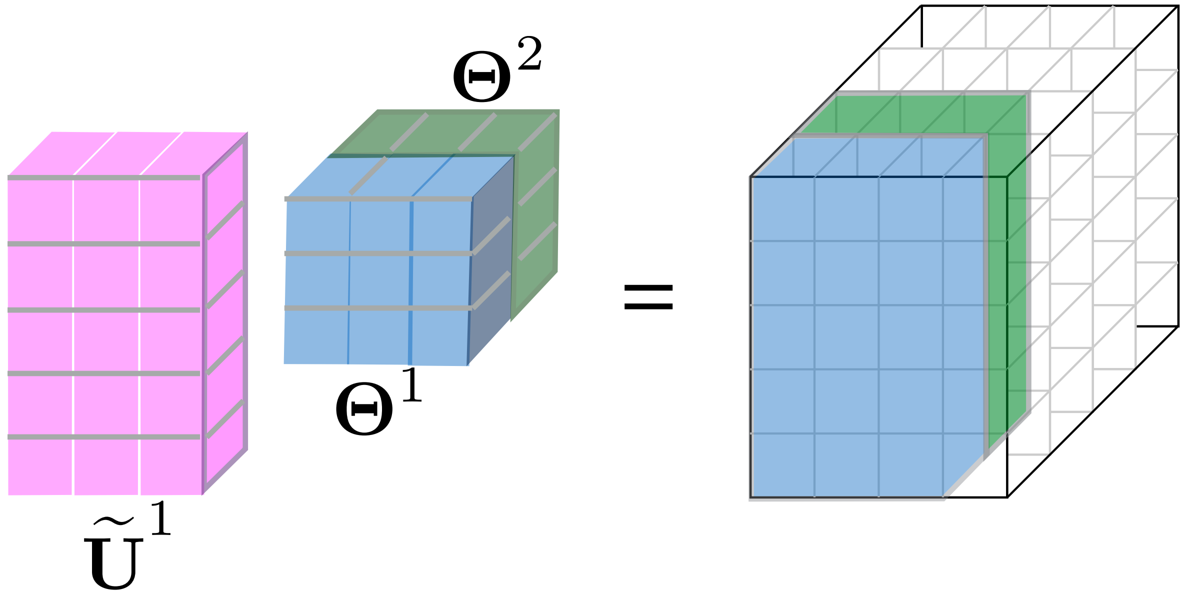


$$\widetilde{\mathbf{U}}^1 = R \text{ l.i. columns in } \mathcal{X}.$$

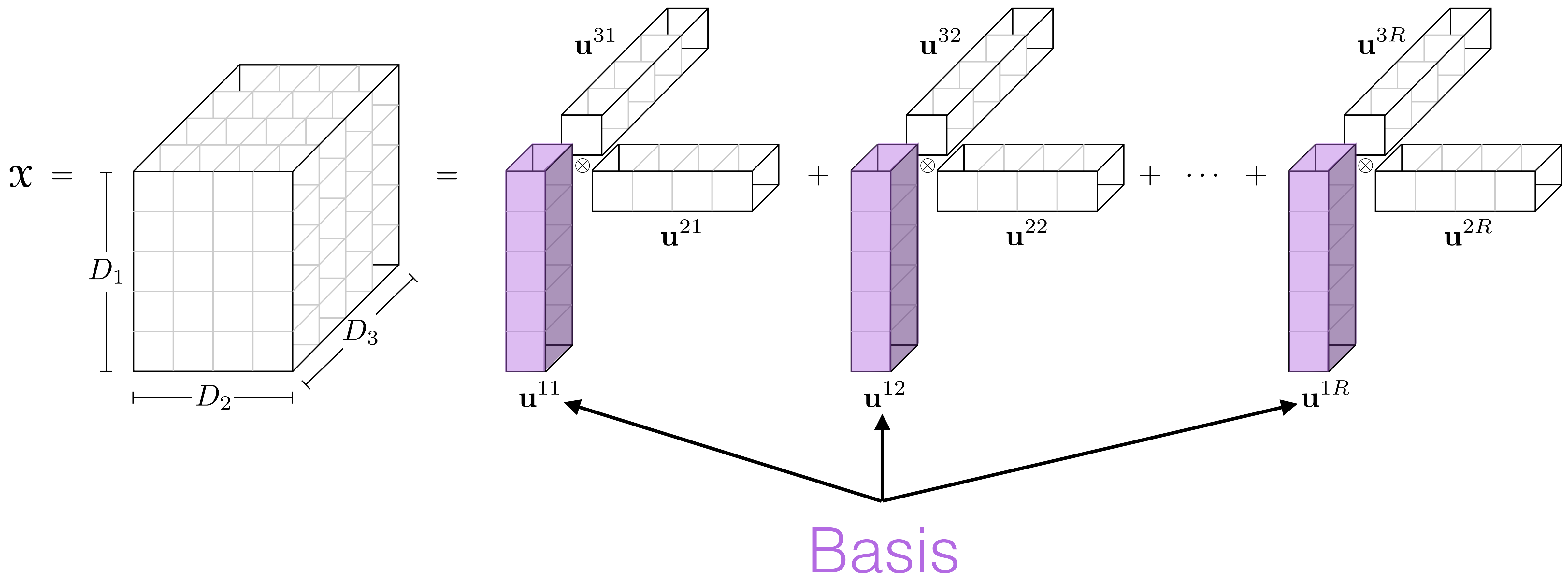
$$\mathbf{\Theta}^1, \mathbf{\Theta}^2 = \text{coeffs of two slices w.r.t. } \widetilde{\mathbf{U}}^1.$$

$$\mathbf{\Gamma} = \text{eigenvectors of } \mathbf{\Theta}^1(\mathbf{\Theta}^2)^{-1}.$$

$$\mathbf{U}^1 = \widetilde{\mathbf{U}}^1\mathbf{\Gamma}.$$

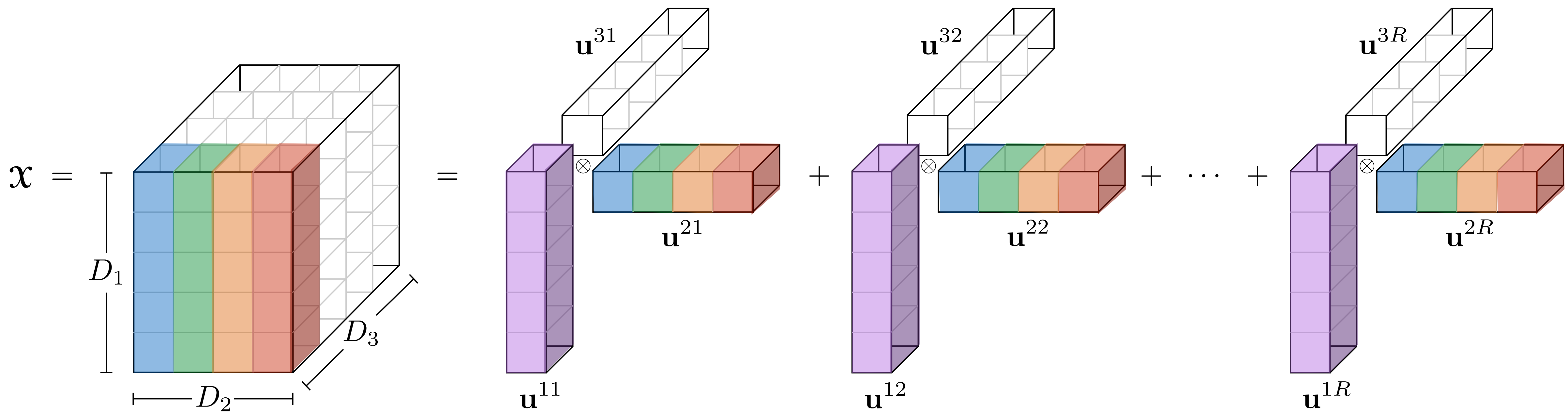
Our Strategy

Summary



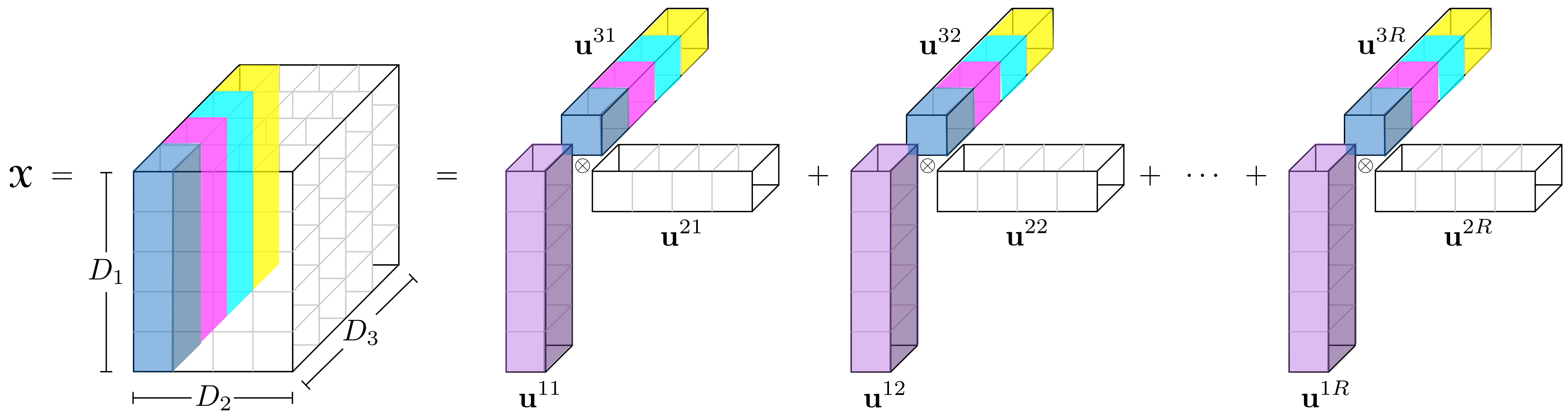
At this point we know the basis

Now we just need to get the coefficients



Our strategy

Take one slice in each dimension



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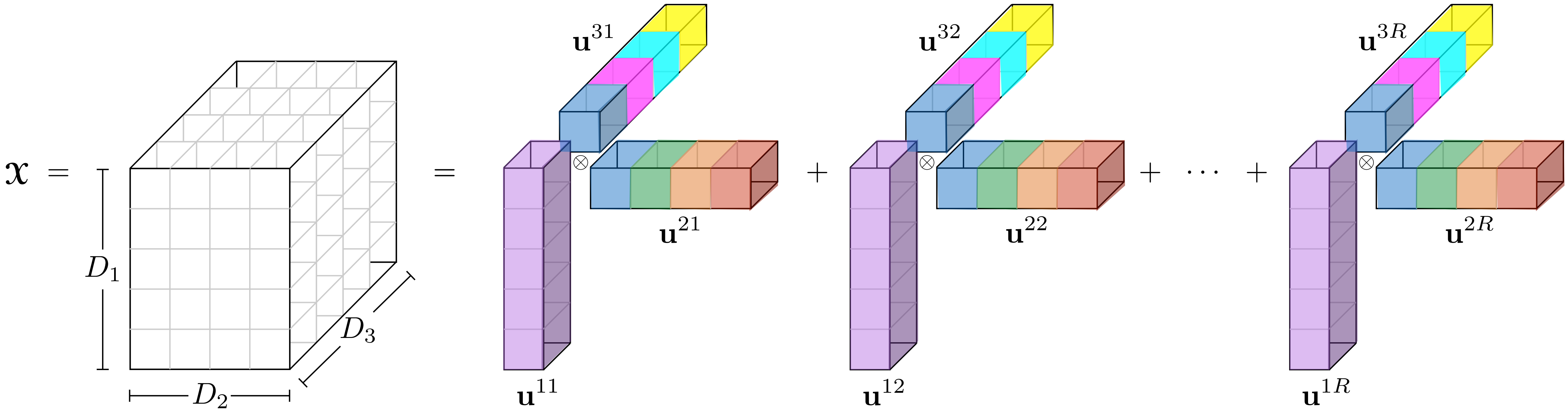
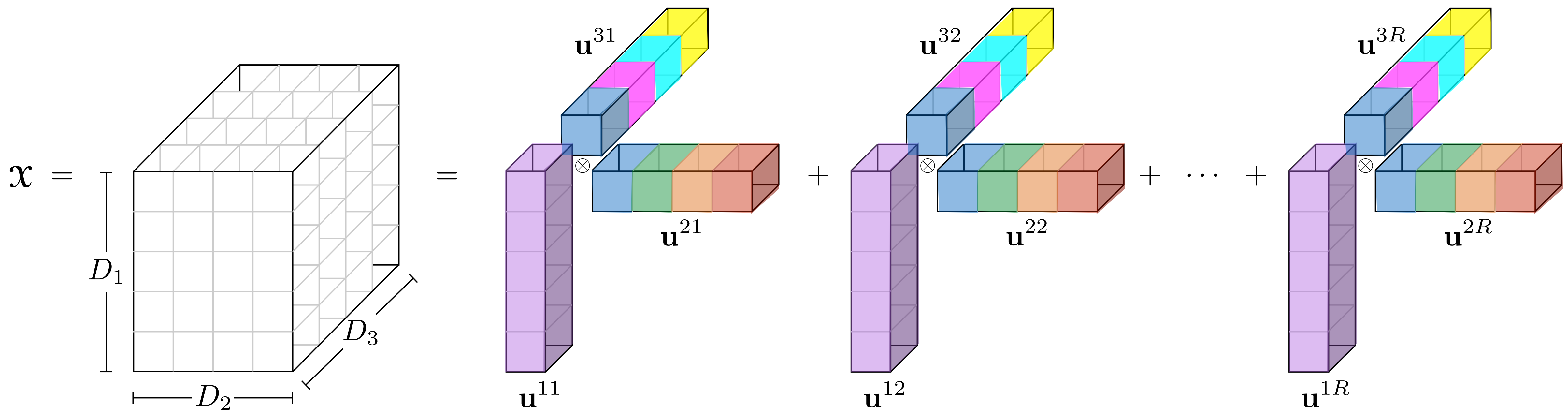


And we are all set!



Take-home message:

We can obtain the CPD of low-rank tensors in *closed form*

What’s next?

What’s next?

Improve method for noisy settings

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$$\mathcal{X} = \sum_{r=1}^{R} \bigotimes_{k=1}^{K} \mathbf{u}^{kr} + \mathcal{W},$$

Improve method for noisy settings

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Our NOISELESS strategy:

Improve method for noisy settings

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Our NOISELESS strategy:

$$\mathbf{U}^1 = \widetilde{\mathbf{U}}^1 \boldsymbol{\Gamma}$$

Improve method for noisy settings

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Can find with SVD

Improve method for noisy settings

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Improve method for noisy settings

Experiments

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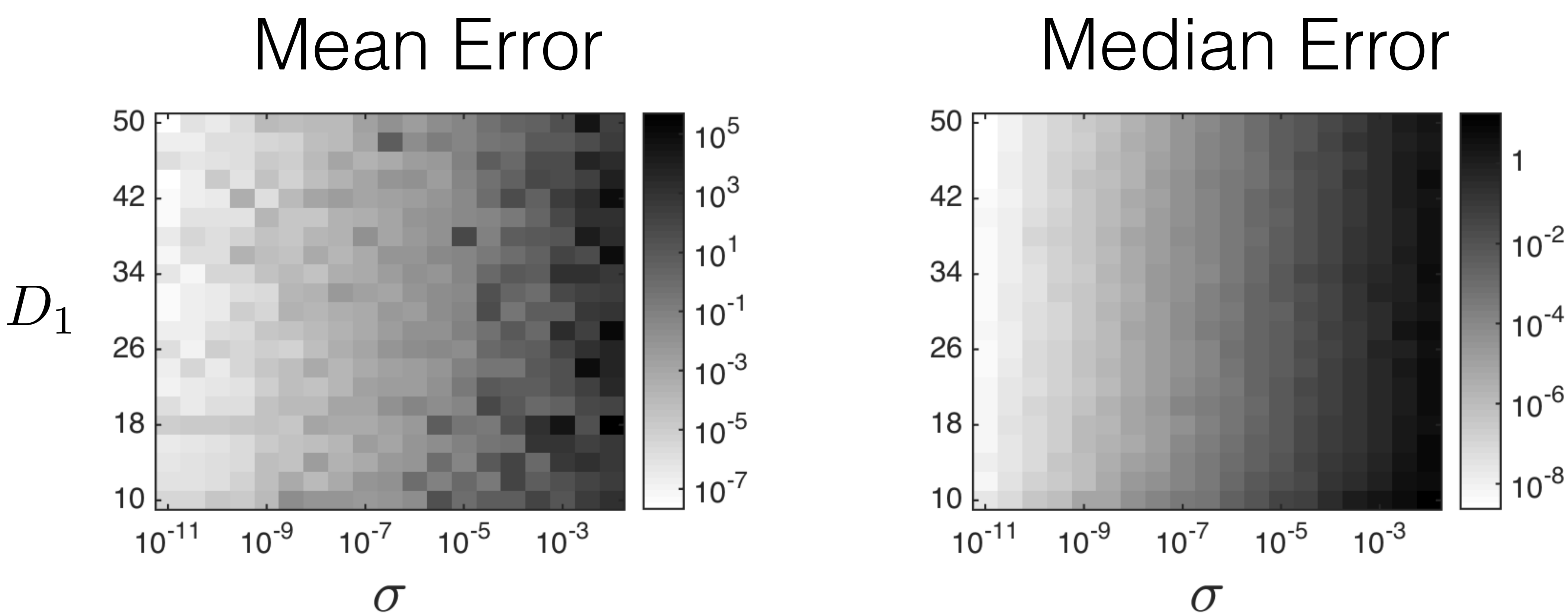

Thank you.