Learning Subspaces by Pieces

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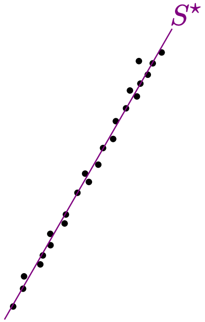
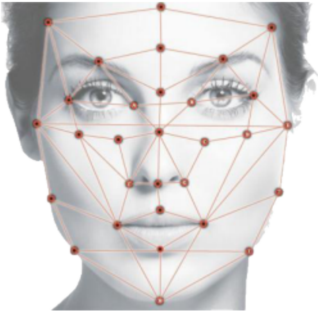
In many Applications we want to Learn Subspaces

$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$



In many Applications we want to Learn Subspaces



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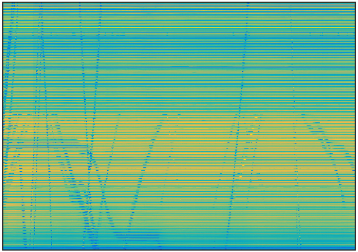

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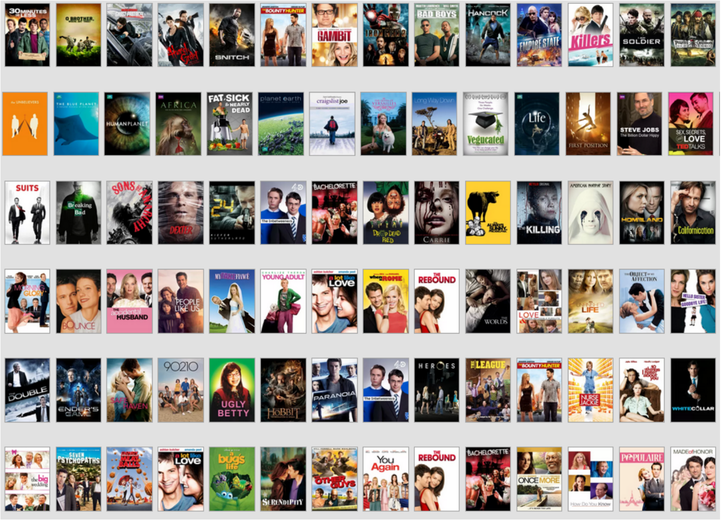

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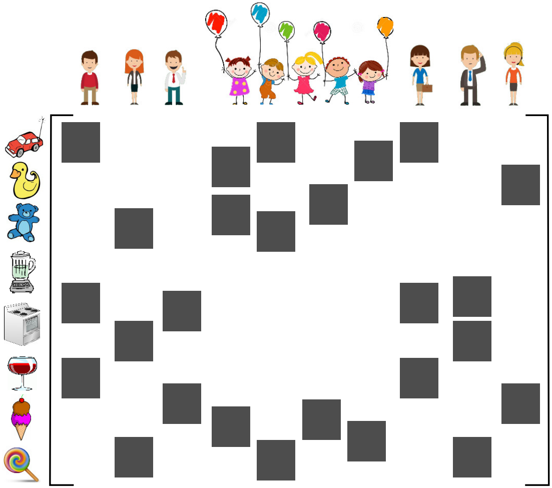
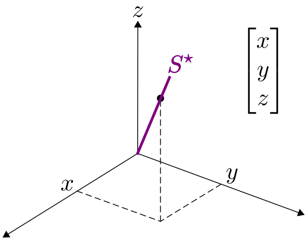


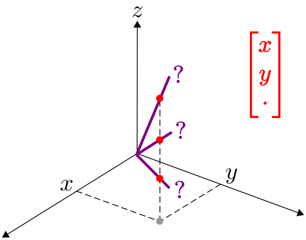
We need to Learn Subspaces by Pieces

$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$

$$\mathbf{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

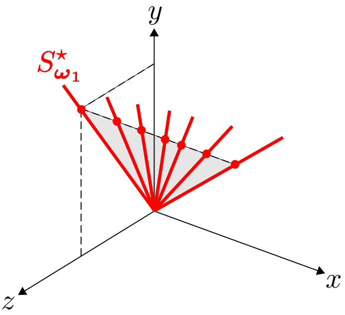


$$\begin{bmatrix} x \\ y \\ \cdot \end{bmatrix}$$

Learning Subspaces by Pieces

A column with $r+1$ observations imposes one restriction on what $S^\star$ may be.

$$\begin{array}{c}\boldsymbol{x}_{\boldsymbol{\omega}_1}\\ \mathbf{X}_{\mathbf{\Omega}} = \left[\begin{array}{c}\cdot\\ 1\\ 1\end{array}\right]\end{array}$$



- A subspace S *agrees* with $\boldsymbol{x}_{\boldsymbol{\omega}_1} \iff \underbrace{f_1(S) = 0}_{\text{degree-}r\text{ polynomial}}$.

Learning Subspaces by Pieces

More precisely:

- Take a basis of S:

$$S \;=\; \operatorname{span} \left. \underbrace{\left[\begin{array}{c} \\ \\ \mathbf{U} \\ \\ \\ \end{array} \right]}_{r} \right\} d.$$

- Then $\boldsymbol{x}_{\boldsymbol{\omega}_i} \in S$ is equivalent to:

$$r+1 \left\{ \left[\begin{array}{c} \\ \boldsymbol{x}_{\boldsymbol{\omega}_i} \\ \\ \end{array} \right] \right. = \left[\begin{array}{c} \\ \mathbf{U}_{\boldsymbol{\omega}_i} \\ \\ \end{array} \right] \boldsymbol{\theta_i}.$$

Learning Subspaces by Pieces

- We can split this as:

$$\begin{array}{r} r \left\{ \vphantom{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} \right. \\ 1 \left\{ \vphantom{a} \right. \end{array} \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{\Delta}_i} \\ \hline \boldsymbol{x}_{\boldsymbol{\nabla}_i} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_{\boldsymbol{\Delta}_i} \\ \hline \mathbf{U}_{\boldsymbol{\nabla}_i} \end{bmatrix} \boldsymbol{\theta}_i.$$

- We can use the top block to solve for $\boldsymbol{\theta}_i$:

$$\boldsymbol{\theta}_i = \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}.$$

- Plug this in the last row:

$$\boldsymbol{x}_{\boldsymbol{\nabla}_i} = \mathbf{U}_{\boldsymbol{\nabla}_i} \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}.$$

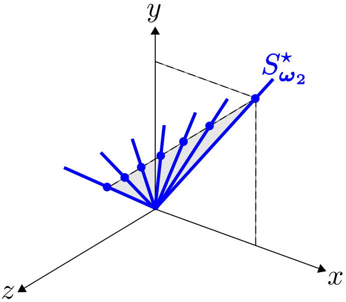
- Or equivalently

$$\underbrace{\boldsymbol{x}_{\boldsymbol{\nabla}_i} - \mathbf{U}_{\boldsymbol{\nabla}_i} \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}}_{f_i(\mathbf{U}_{\boldsymbol{\omega}_i} | \boldsymbol{x}_{\boldsymbol{\omega}_i})} = 0.$$

Learning Subspaces by Pieces

An other column with $r+1$ samples imposes an other restriction.

$$\mathbf{X}_{\boldsymbol{\Omega}} = \begin{array}{c} \boldsymbol{x}_{\boldsymbol{\omega}_2} \\ \left[\begin{array}{c} 2 \\ 2 \\ \cdot \end{array}\right] \end{array}$$

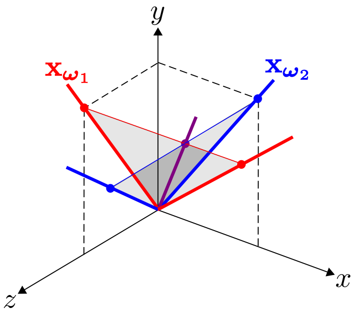


- A subspace S *agrees* with $\boldsymbol{x}_{\boldsymbol{\omega}_2} \iff f_2(\mathbf{U}_{\boldsymbol{\omega}_2} | \boldsymbol{x}_{\boldsymbol{\omega}_2}) = 0$.

Learning Subspaces by Pieces

Each column with $r+1$ samples imposes one restriction.

$$\mathbf{X}_{\boldsymbol{\Omega}} \quad = \quad \begin{array}{c} \begin{array}{cc} \boldsymbol{x}_{\boldsymbol{\omega}_1} & \boldsymbol{x}_{\boldsymbol{\omega}_2} \end{array} \\ \left[\begin{array}{cc} \cdot & 2 \\ 1 & 2 \\ 1 & \cdot \end{array} \right] \end{array}$$



- A subspace S *agrees* with $\mathbf{X}_{\boldsymbol{\Omega}} \iff \left\{ \begin{array}{l} f_1(\mathbf{U}_{\boldsymbol{\omega}_1} | \boldsymbol{x}_{\boldsymbol{\omega}_1}) = 0 \\ f_2(\mathbf{U}_{\boldsymbol{\omega}_2} | \boldsymbol{x}_{\boldsymbol{\omega}_2}) = 0 \end{array} \right.$.

Learning Subspaces by Pieces

- We thus obtain a set of *generic* polynomials:

$$f_1(\mathbf{U}_{\boldsymbol{\omega}_1}), f_2(\mathbf{U}_{\boldsymbol{\omega}_2}), \ldots, f_N(\mathbf{U}_{\boldsymbol{\omega}_N}).$$

- Polynomial f_i only involves the variables indicated in $\boldsymbol{\omega}_i$.
- Construct $\boldsymbol{\Omega} = [\boldsymbol{\omega}_1 \ \boldsymbol{\omega}_2 \ \cdots \ \boldsymbol{\omega}_N]$.
  - Each column of $\boldsymbol{\Omega}$ corresponds to one polynomial.
  - Its nonzero rows indicate the variables involved.

Learning Subspaces by Pieces

- Polynomials are a algebraically independent iff

$$\underbrace{n(\boldsymbol{\Omega}')}_{equations} \leq \underbrace{r(m(\boldsymbol{\Omega}') - r)}_{unknowns} \qquad \forall\ \boldsymbol{\Omega}' \subset \boldsymbol{\Omega}.$$

After this, deep algebraic geometry results do the heavy lifting:

$\Leftrightarrow$ Polynomials are a regular sequence.

$\Leftrightarrow$ Polynomials define a zero-dimensional variety.

$\Leftrightarrow$ At most finitely many solutions (subspaces) will agree with $\mathbf{X}_{\boldsymbol{\Omega}}$.

Learning Subspaces by Pieces

Theorem (P.-A., Boston, Nowak)

For almost every $\mathbf{X}$, at most finitely many r-dimensional subspaces can agree with $\mathbf{X}_{\mathbf{\Omega}}$ if and only if every matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$ satisfies

$$m(\mathbf{\Omega}') \;\geq\; n(\mathbf{\Omega}')/r + r.$$

Learning Subspaces by Pieces

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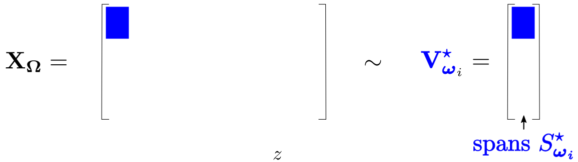
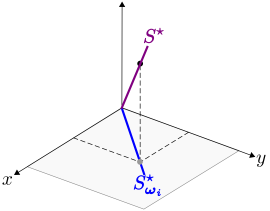
$$\mathbf{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix} \qquad \mathbf{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & 3 & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix}$$

$$\underbrace{m(\mathbf{\Omega}')}_{3} \not\geq \underbrace{n(\mathbf{\Omega}')/r + r}_{4} \qquad \underbrace{m(\mathbf{\Omega}')}_{4} \geq \underbrace{n(\mathbf{\Omega}')/r + r}_{4}$$

Some Pieces are Better than Others

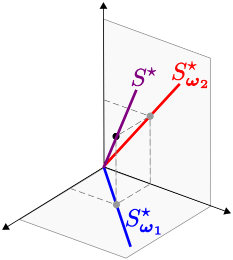
If we observe blocks, then polynomials become linear!

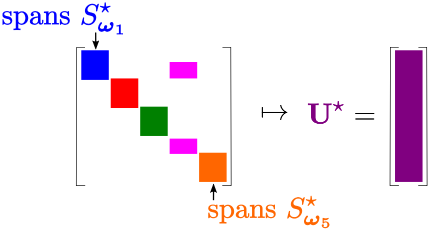




Some Pieces are Better than Others

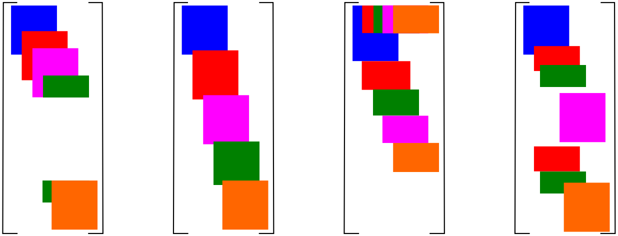
- We are given a bunch of *pieces* of the subspace.
- We want to reconstruct the whole subspace.





Theorem tells us...

- Which pieces to observe.
- How to reconstruct the subspace.

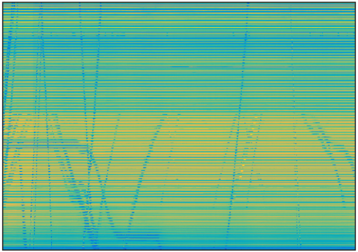


- Now we know which pieces we need.
- And how to reconstruct $S^{\star}$ from its pieces.
- OK, cool, that's all very nice, but...
- This is Applied Algebra. So, what is this good for?

What is this good for?

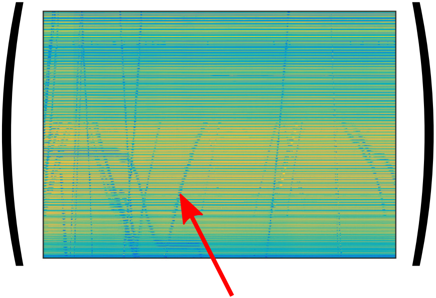
1. Background Segmentation
- If time allows
  2. Clustering
  3. Missing Data

Background Segmentation



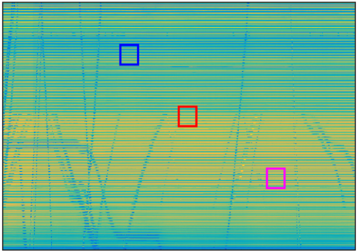
Background Segmentation

SVD

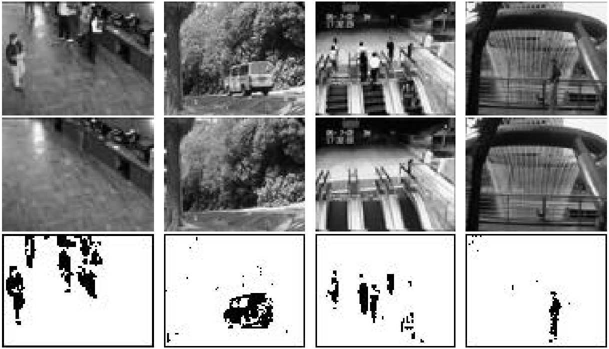


Outliers!

Background Segmentation

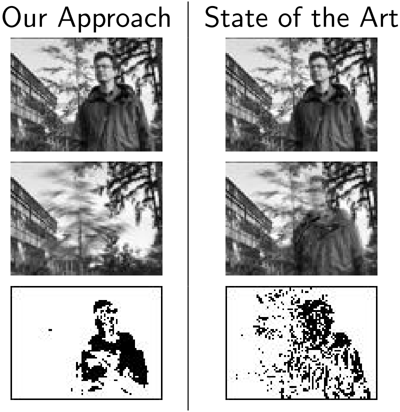


Background Segmentation



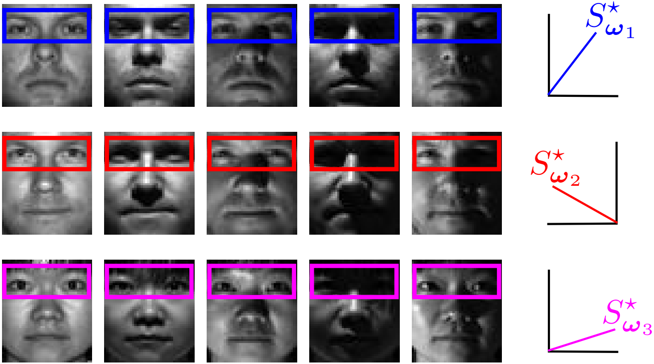
Background Segmentation

Background Segmentation

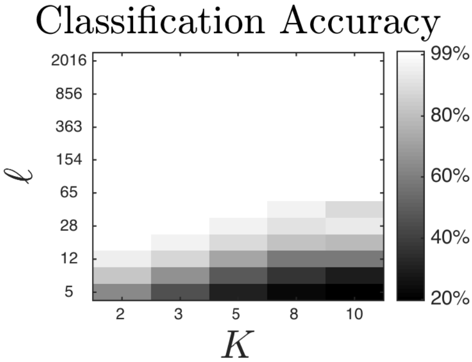


How am I on time?

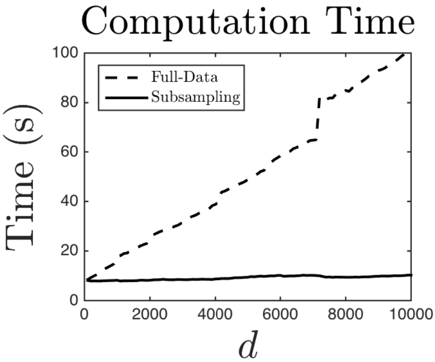
Clustering



Clustering

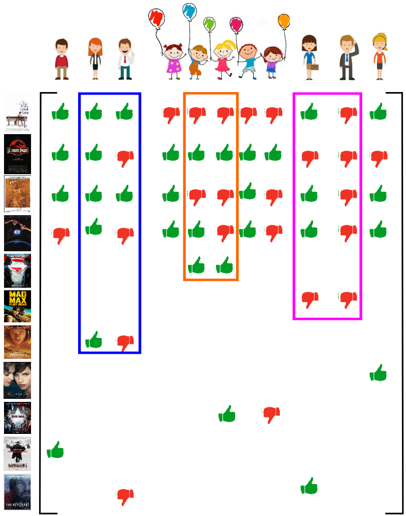

Classification Accuracy
2016
856
363
154
65
28
12
5
ℓ
2
3
5
8
10
K
99%
80%
60%
40%
20%

Clustering



Computation Time
Full-Data
Subsampling
Time (s)
100
80
60
40
20
0
0
2000
4000
6000
8000
10000
d

Missing Data



Thanks.