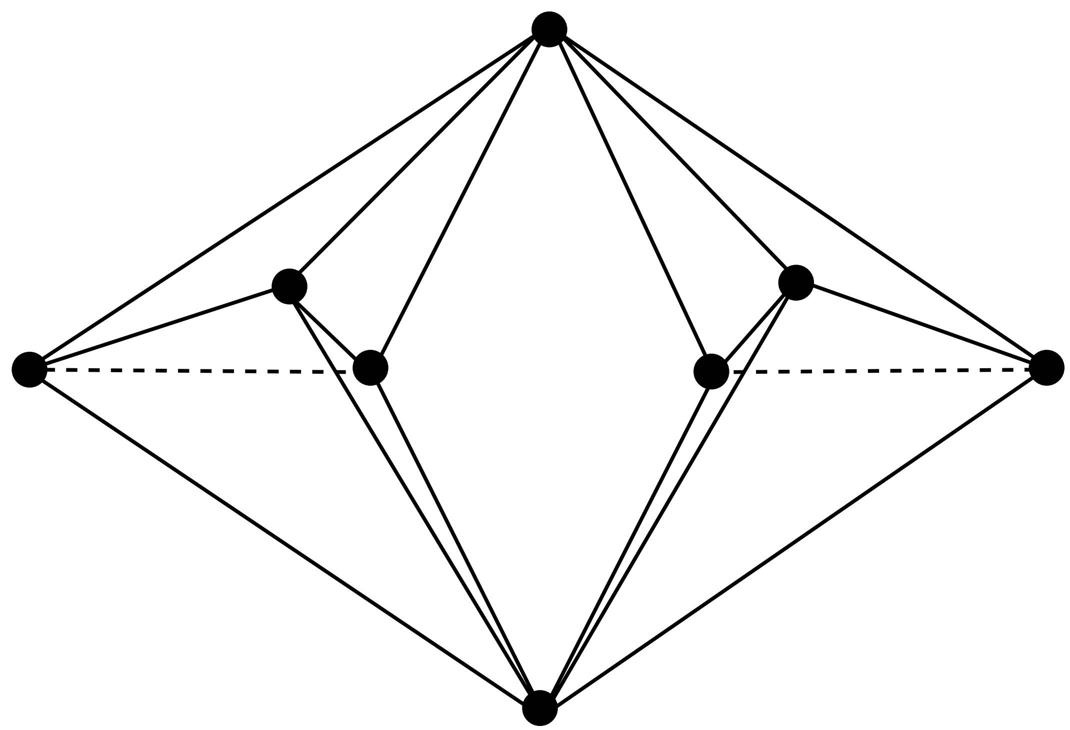
A Converse to Low-Rank Matrix Completion

Daniel Pimentel-Alarcón & Robert Nowak
University of Wisconsin-Madison

Low-Rank Matrices are all around!



$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Sometimes we KNOW

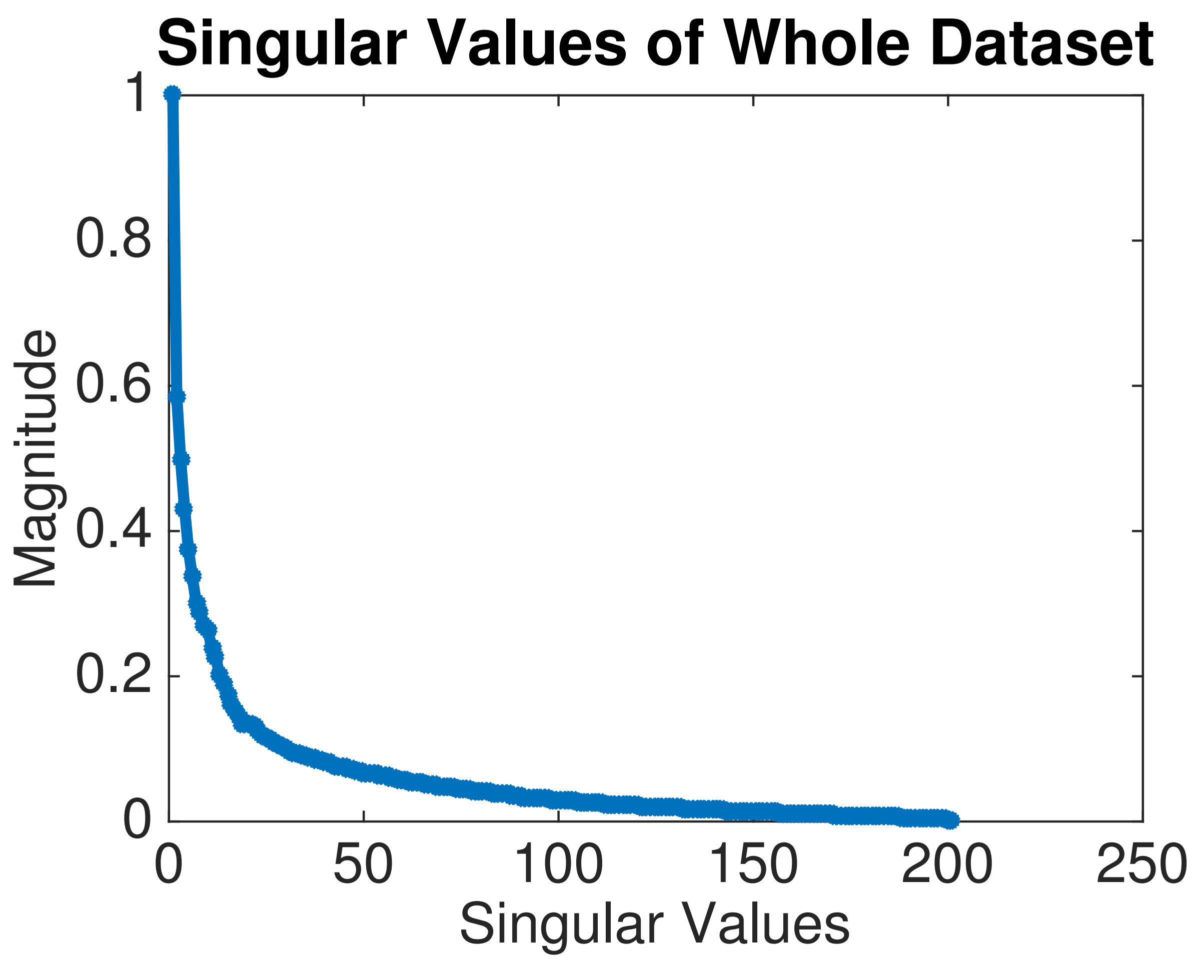
that Matrix is Low-Rank

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$

But in General

We just have a data matrix that we want to analyze

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$



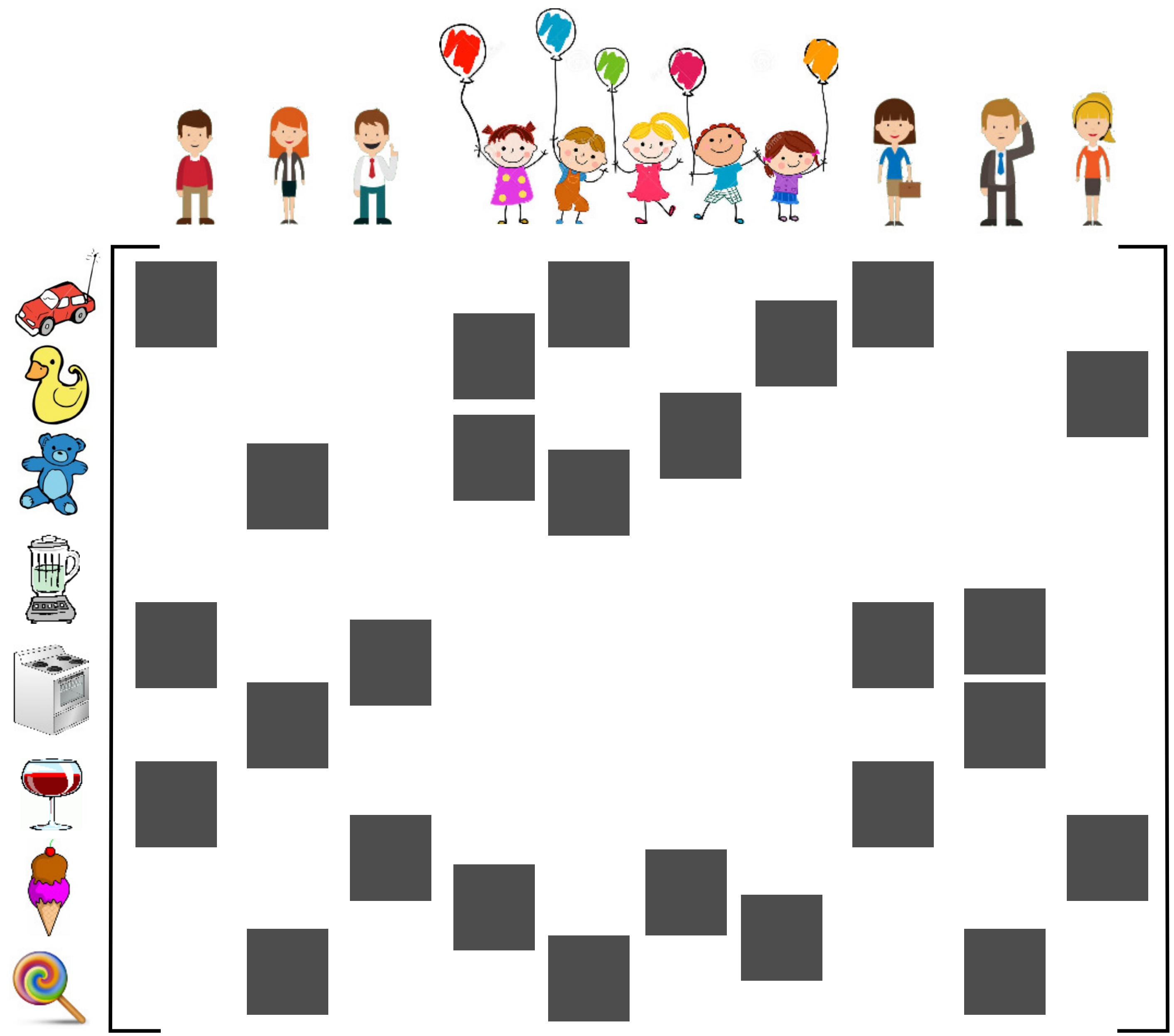
But in General

We just have a data matrix that we want to analyze
Typically we use SVD

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

But if Data is Missing

We CANNOT use SVD!



Sometimes we HOPE

that Matrix is Low-Rank

Just because we can find a rank-*r* completion

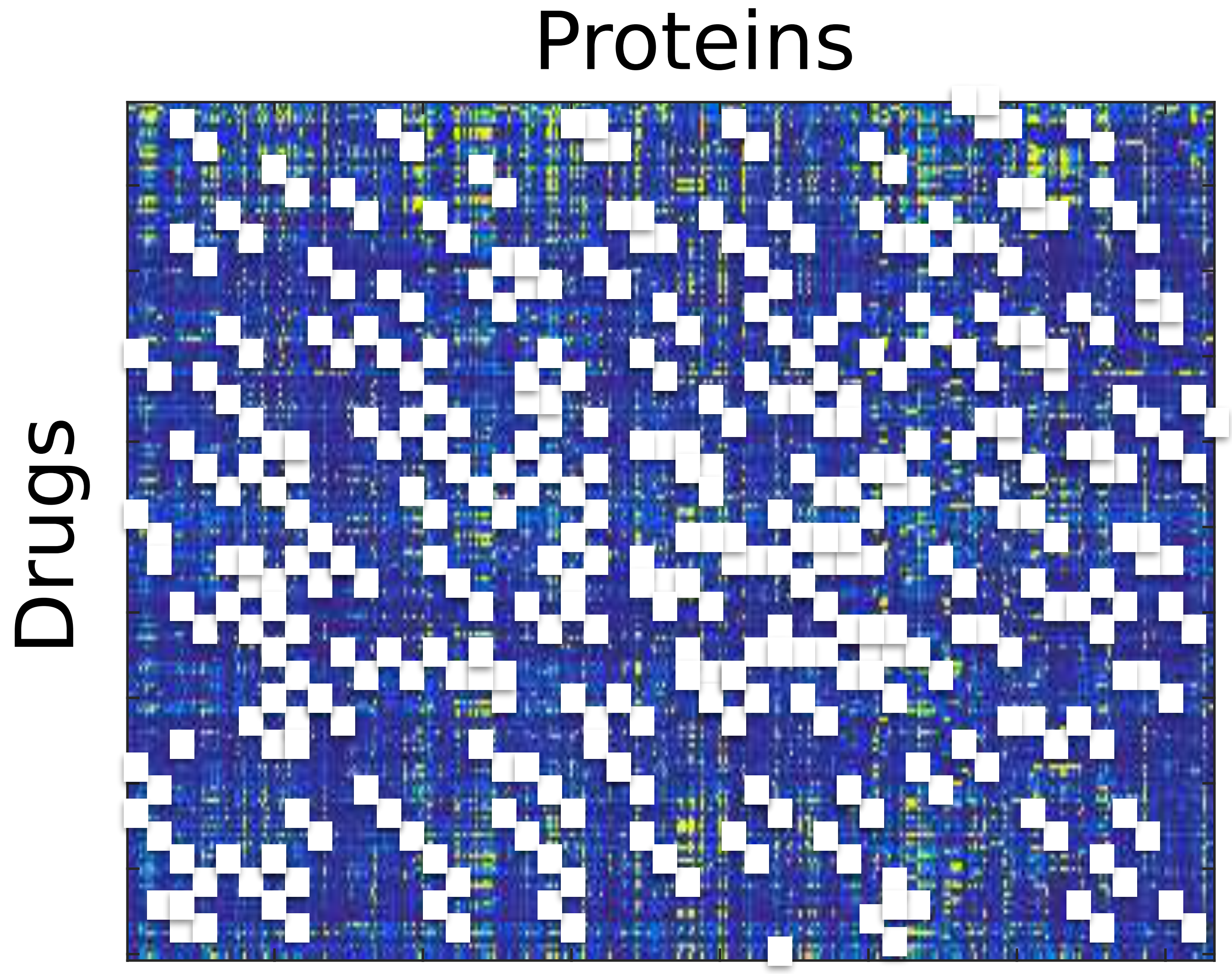
$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$

Doesn't mean the full matrix was rank-*r*

Just because we can find a rank-*r* completion

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & \mathbf{X} & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & \mathbf{X} & 2 & 6 & 3 & 2 & 2 & \mathbf{X} & 4 & \mathbf{X} \\ 3 & 4 & 3 & 9 & 3 & 3 & \mathbf{X} & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & \mathbf{X} & 6 & 2 & 4 & 2 & 4 & \mathbf{X} \\ \mathbf{X} & 8 & 3 & 9 & \mathbf{X} & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$

Doesn't mean the full matrix was rank-*r*

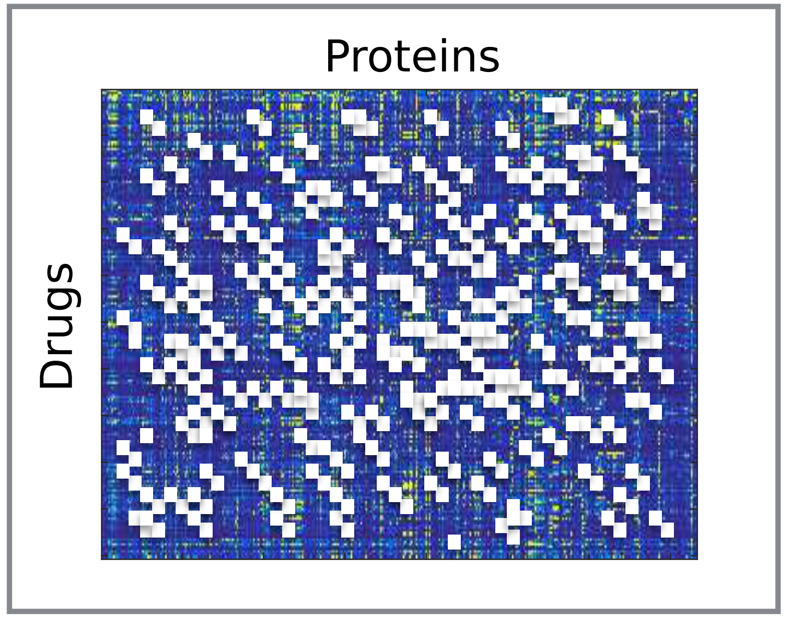


Sometimes, we better MAKE SURE!

that Matrix is Low-Rank

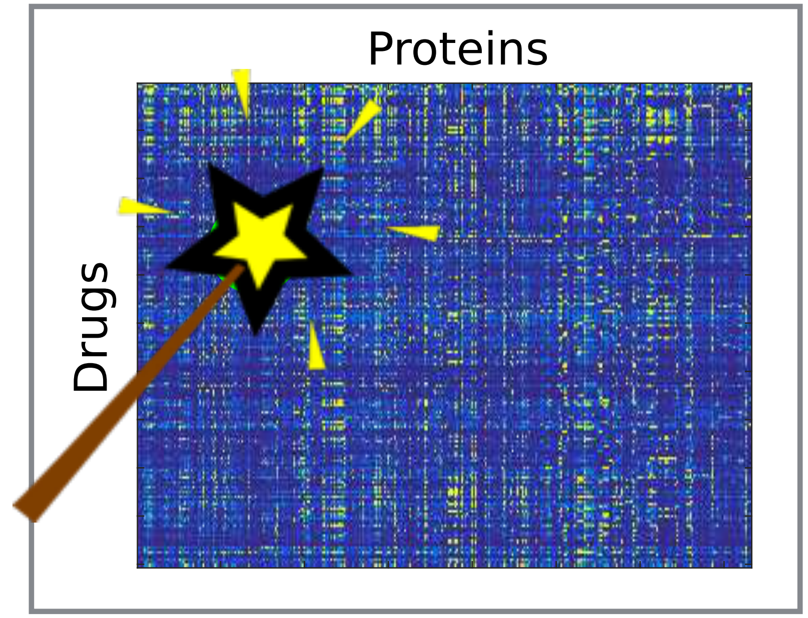
Sometimes, we better MAKE SURE!

that Matrix is Low-Rank



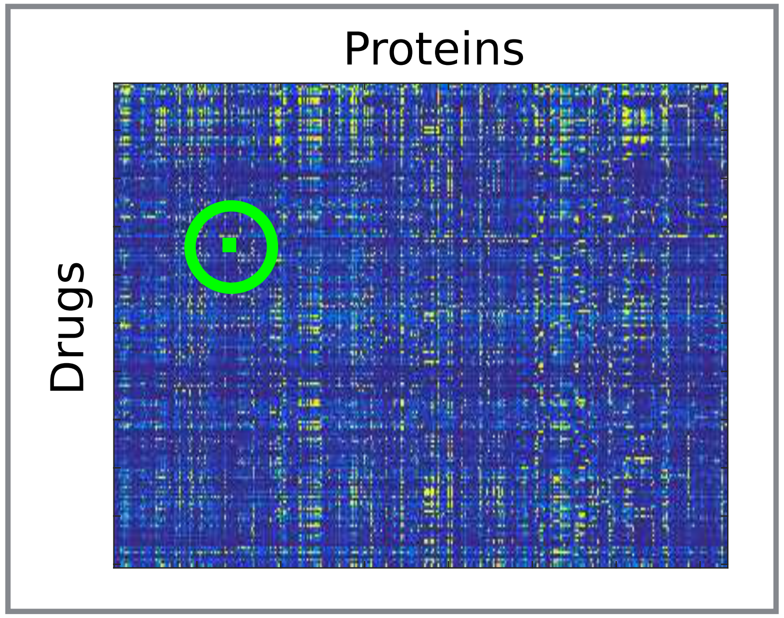
Sometimes, we better MAKE SURE!

that Matrix is Low-Rank



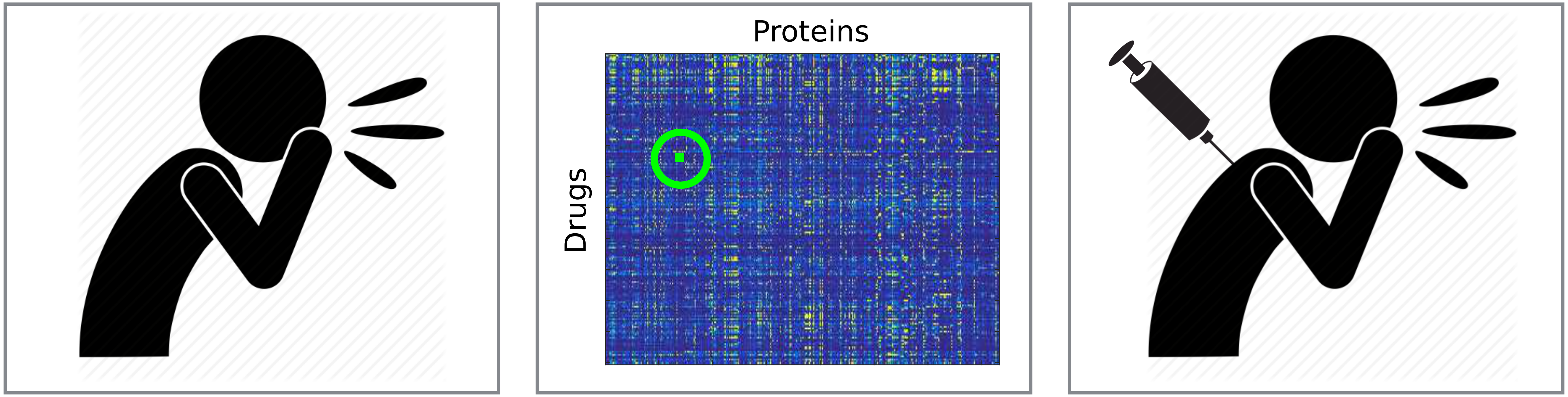
Sometimes, we better MAKE SURE!

that Matrix is Low-Rank



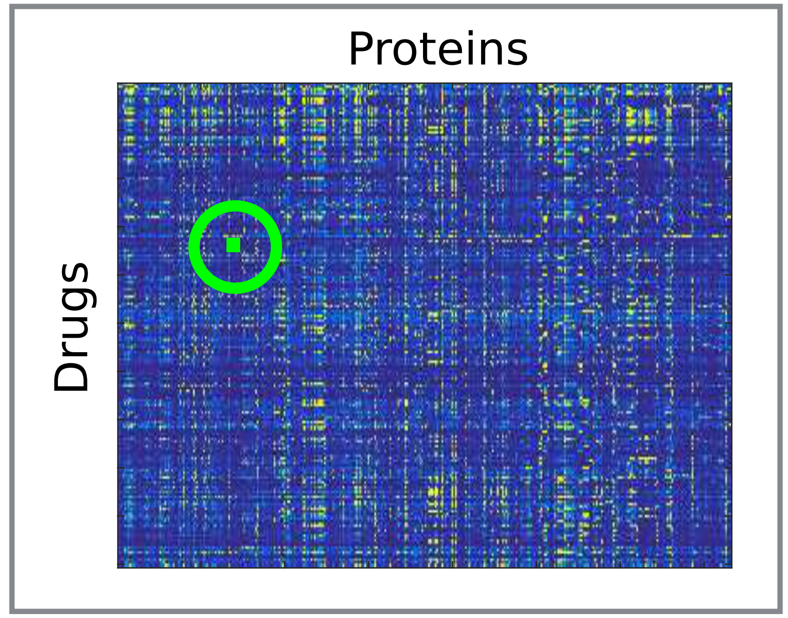
Sometimes, we better MAKE SURE!

that Matrix is Low-Rank



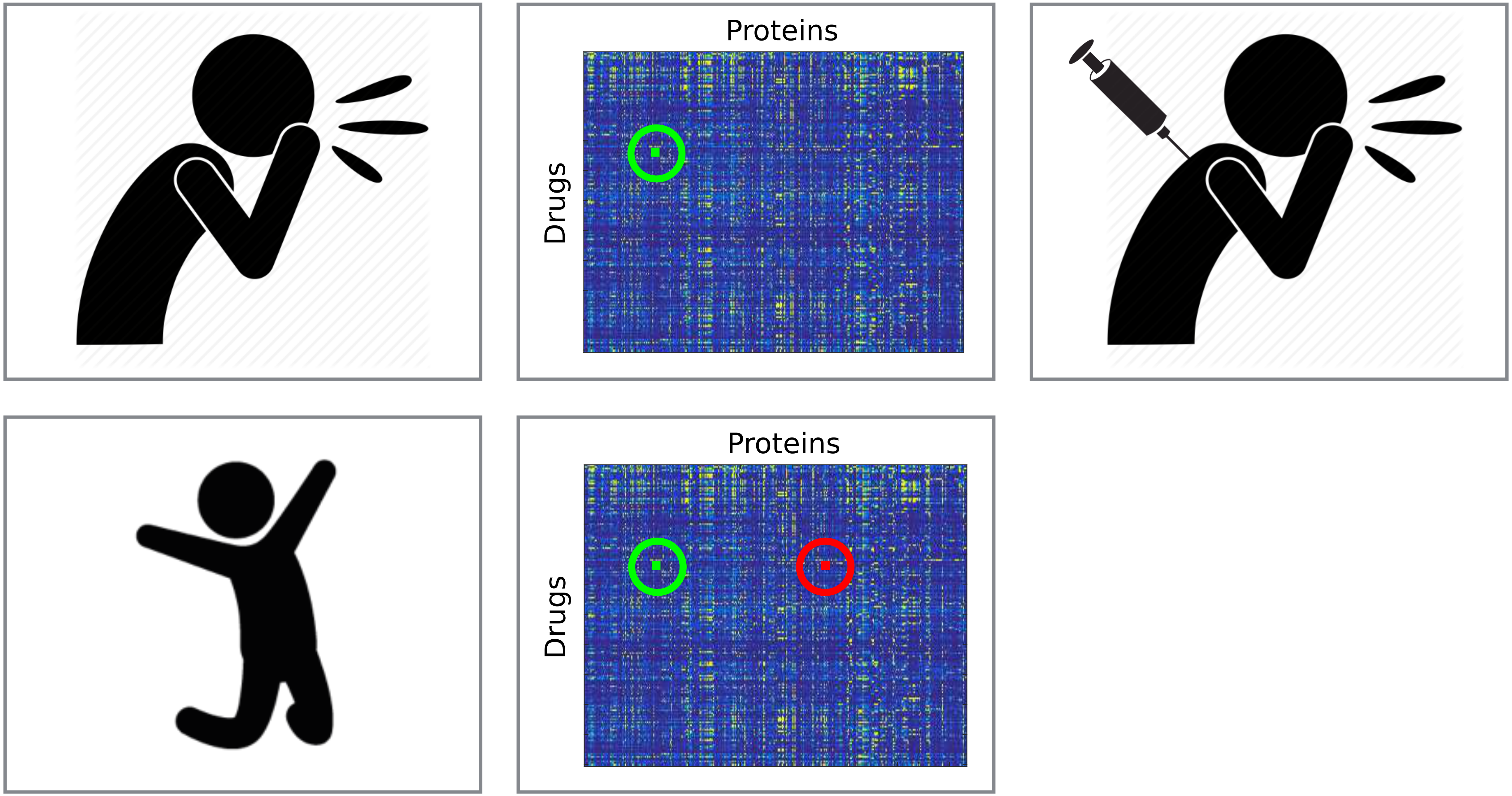
Sometimes, we better MAKE SURE!

that Matrix is Low-Rank



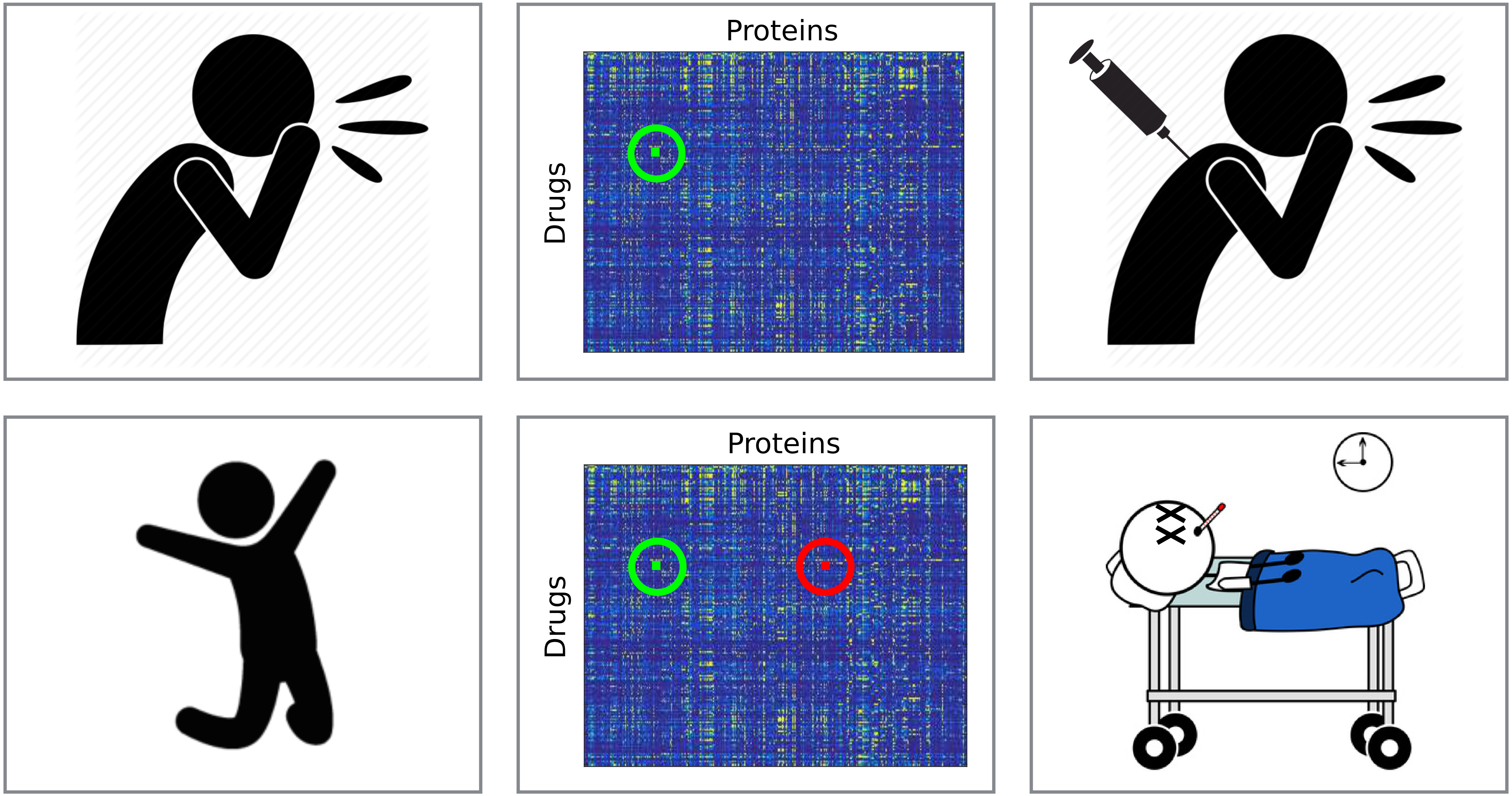
Sometimes, we better MAKE SURE!

that Matrix is Low-Rank



Sometimes, we better MAKE SURE!

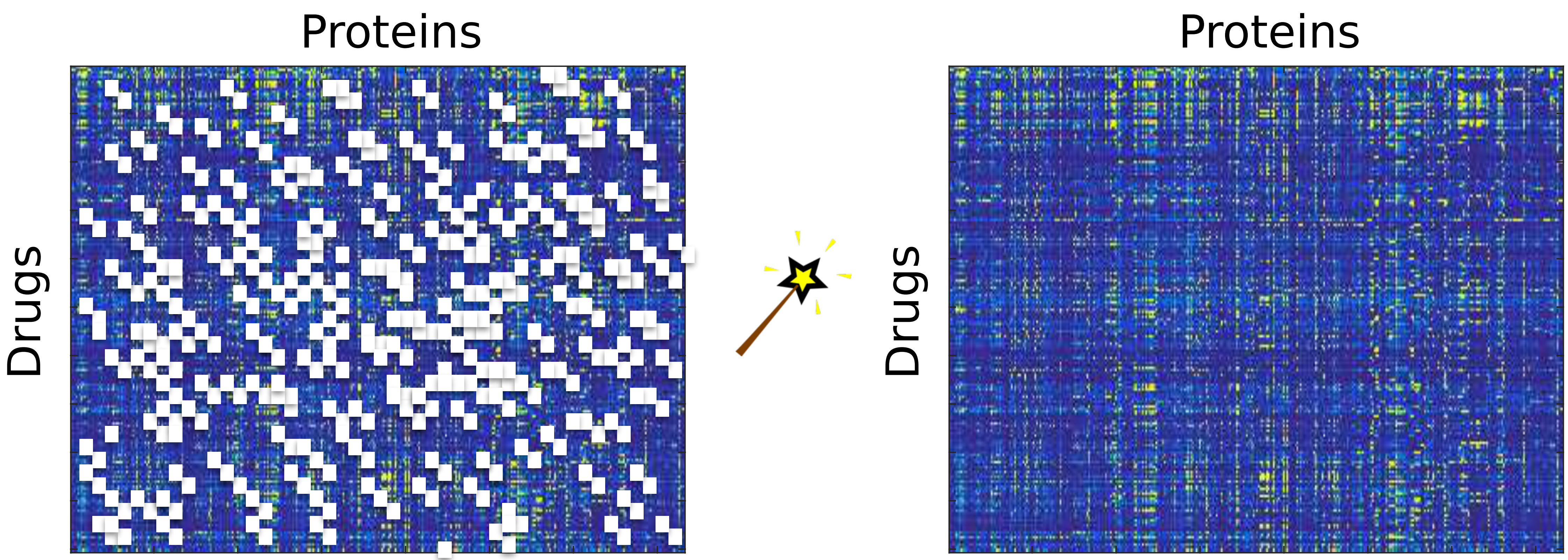
that Matrix is Low-Rank



Sometimes, we better MAKE SURE!

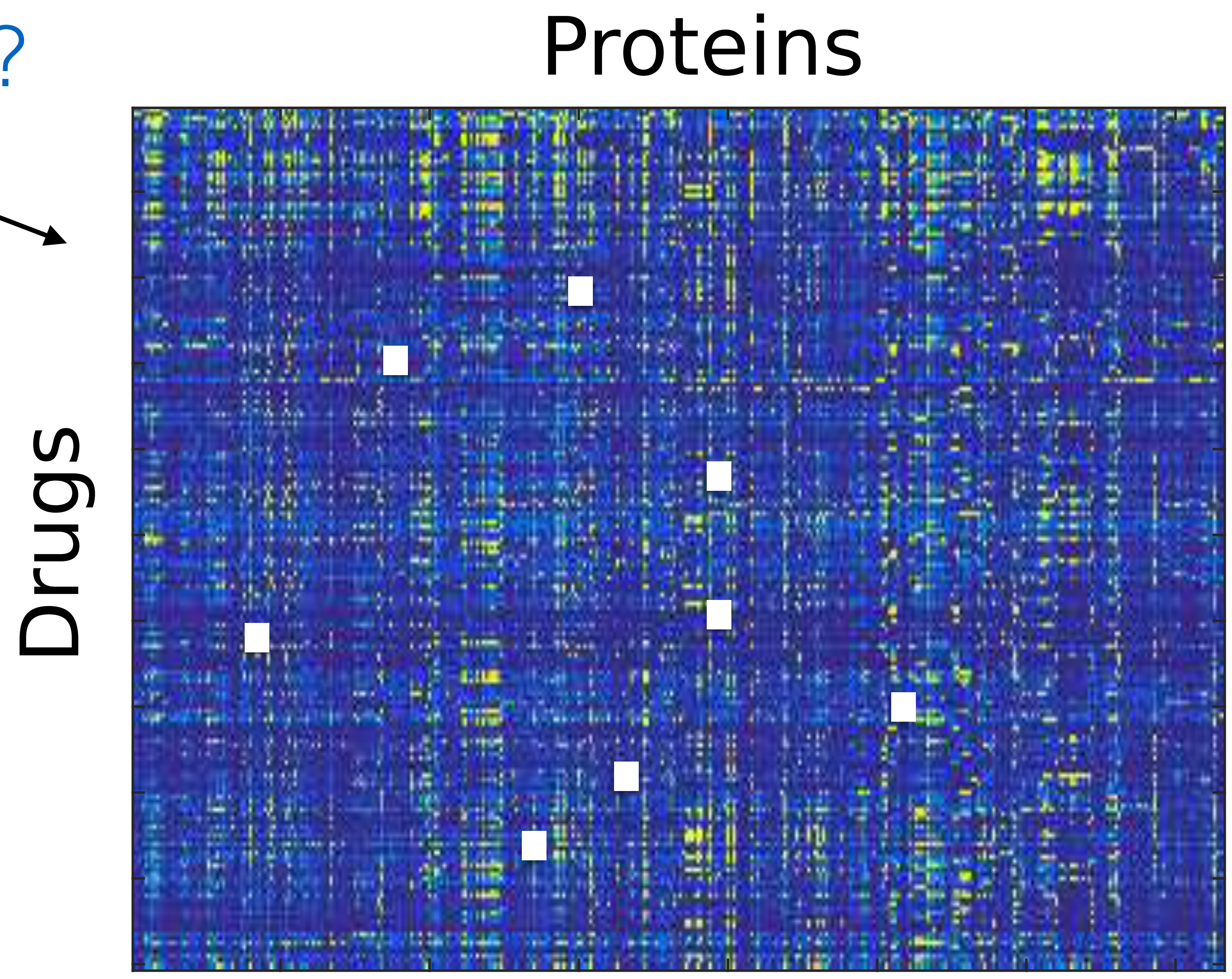
that Matrix is Low-Rank

Say I find a rank-r completion.



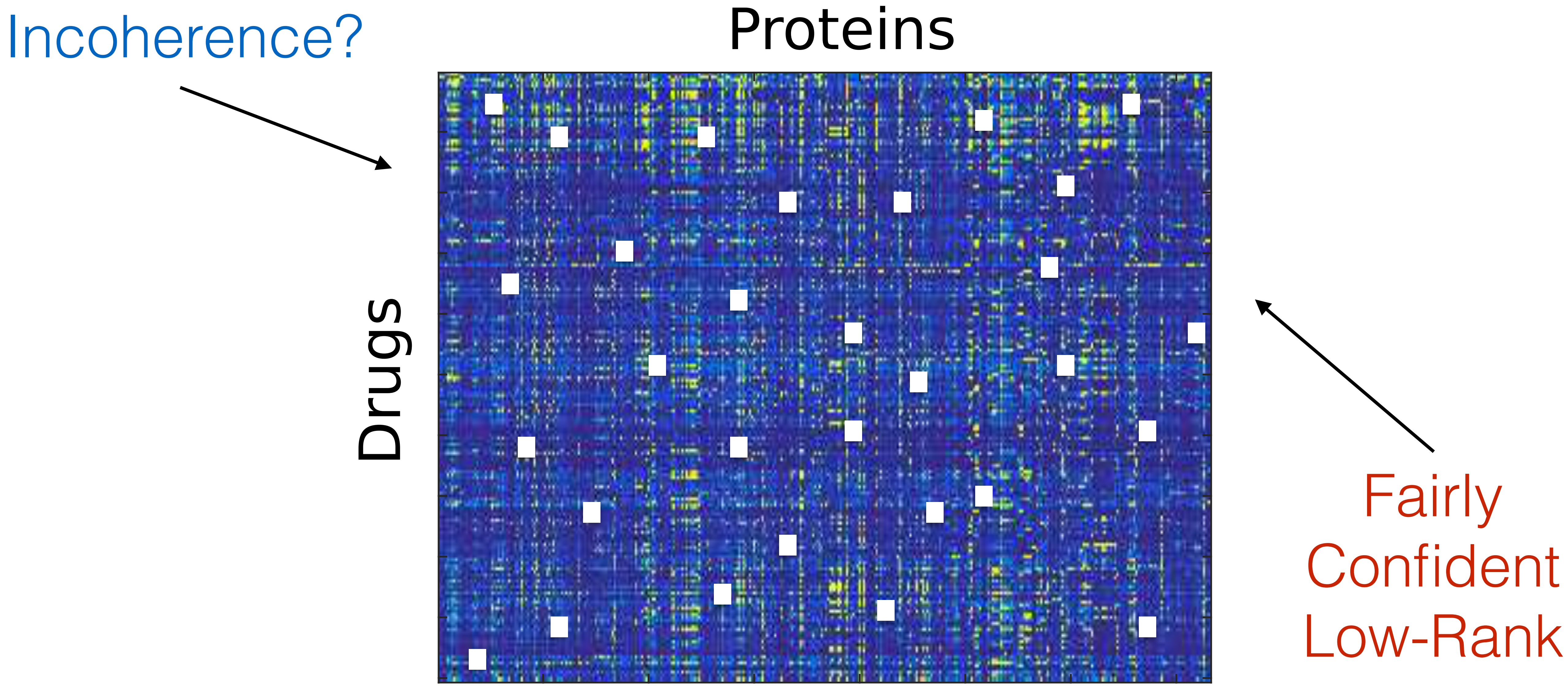
How can I know that
full-matrix was indeed low-rank?

Incoherence?

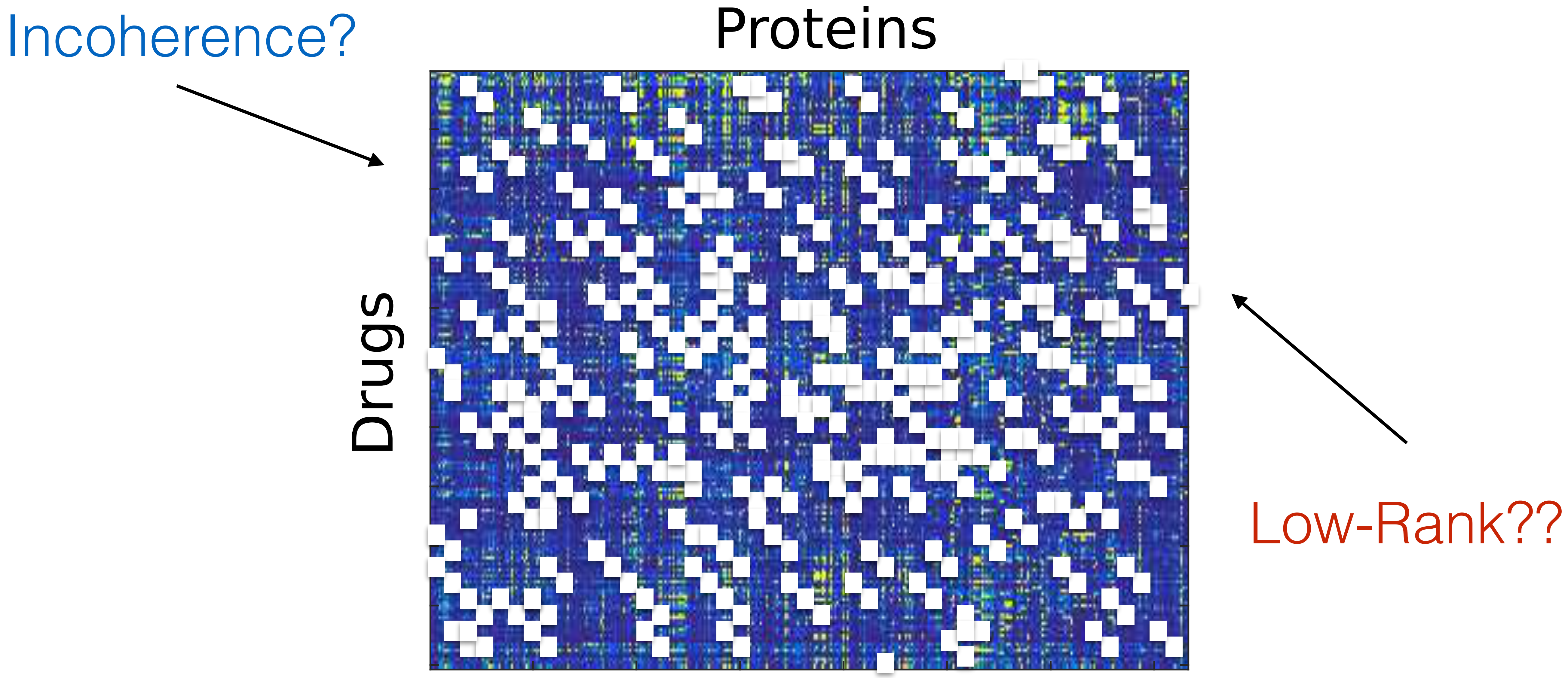


Pretty Confident Low-Rank

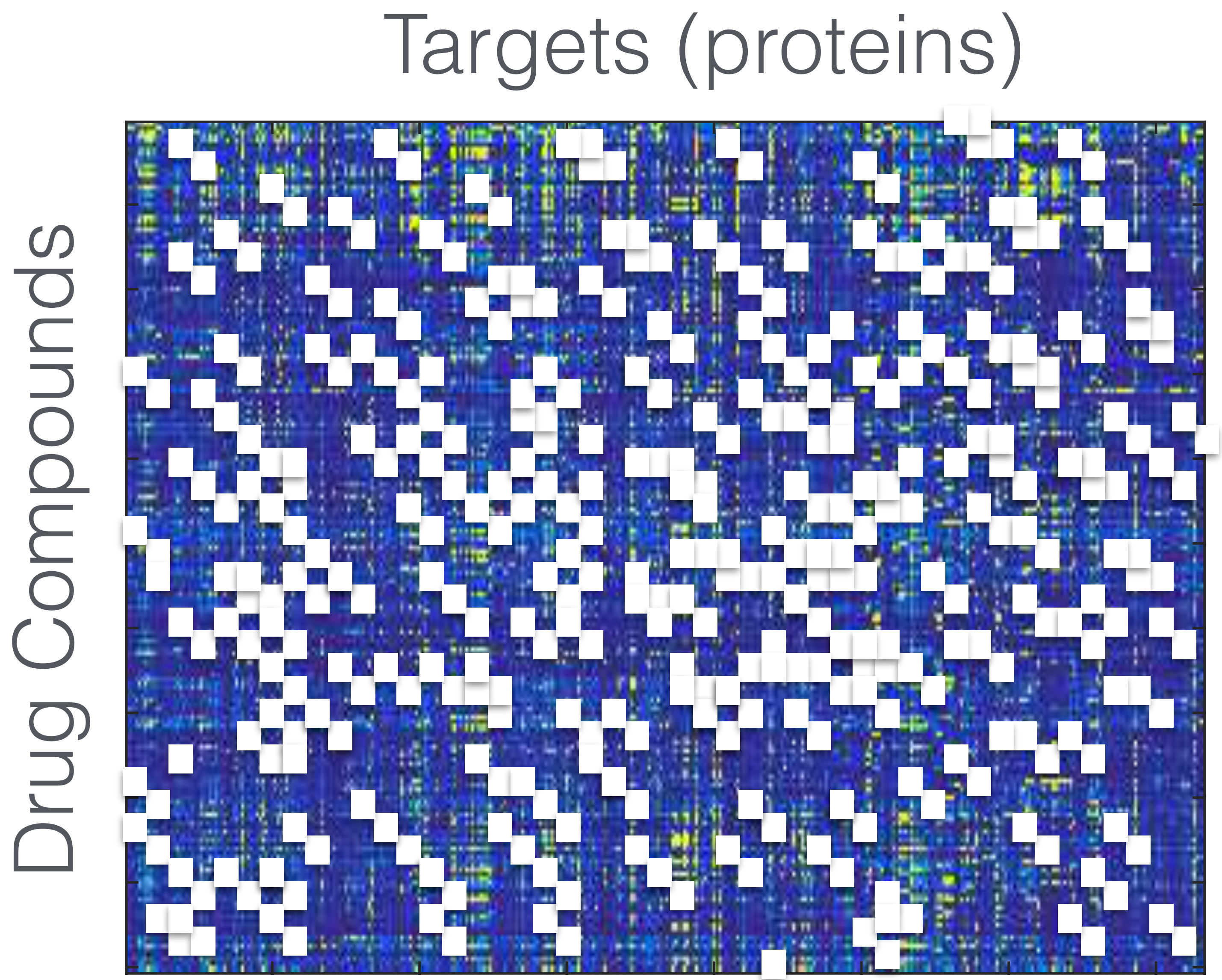
Intuition



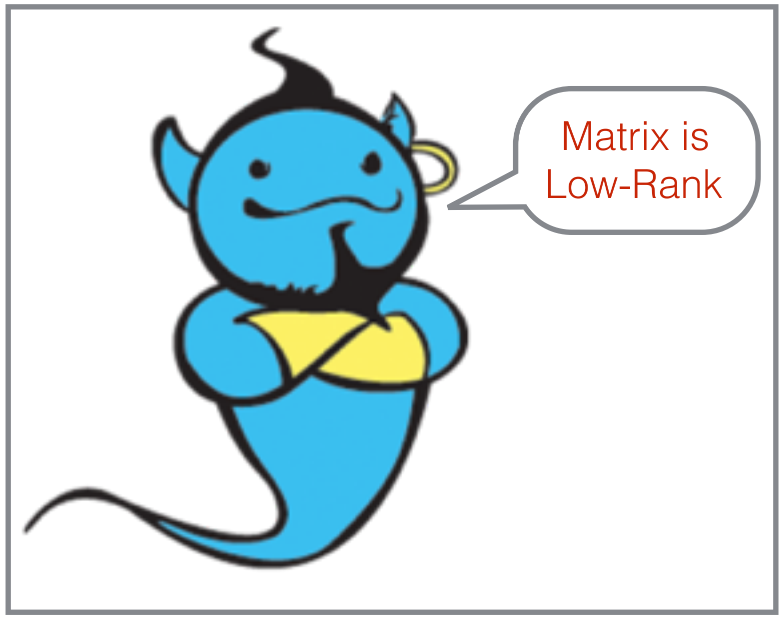
Intuition



Intuition



Intuition

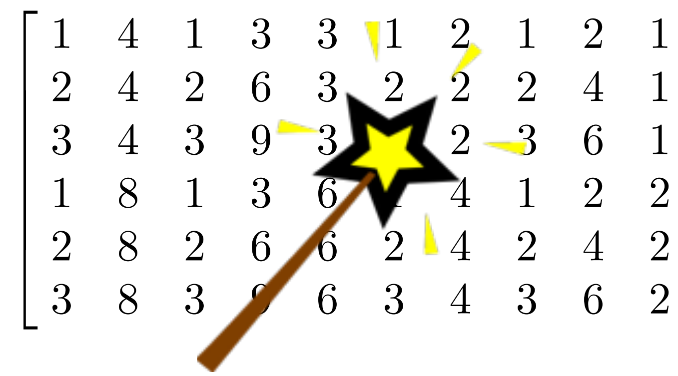


&

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

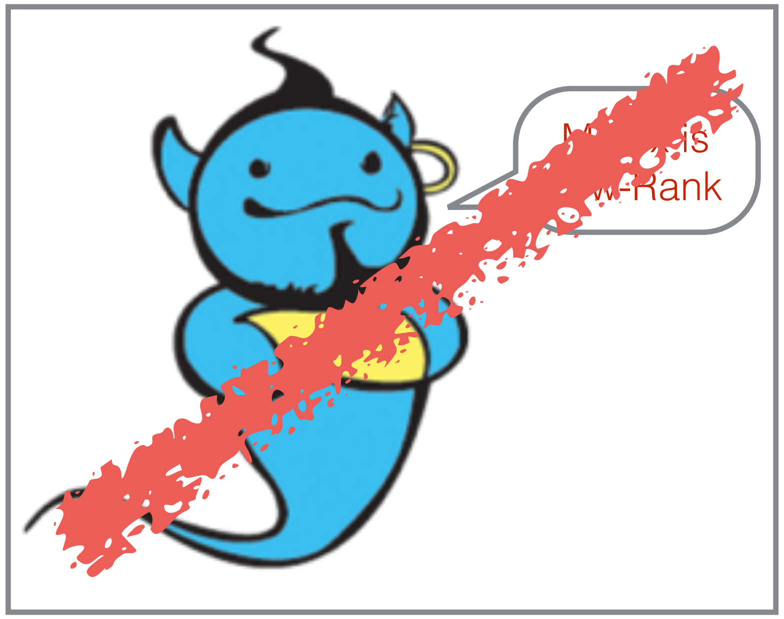
- **Conditions on Matrix**
  (e.g., incoherence)
- **Conditions on samples**
  (e.g., uniformly distributed)

$\Rightarrow$



Matrix can be Uniquely Completed

Existing Theory



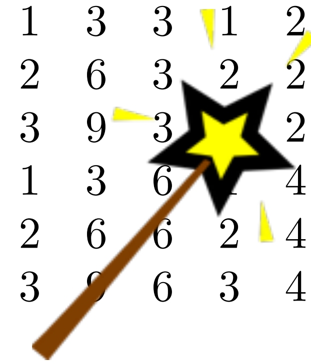
&

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

- **Conditions on Matrix**
  (e.g., incoherence)
- **Conditions on samples**
  (e.g., uniformly distributed)

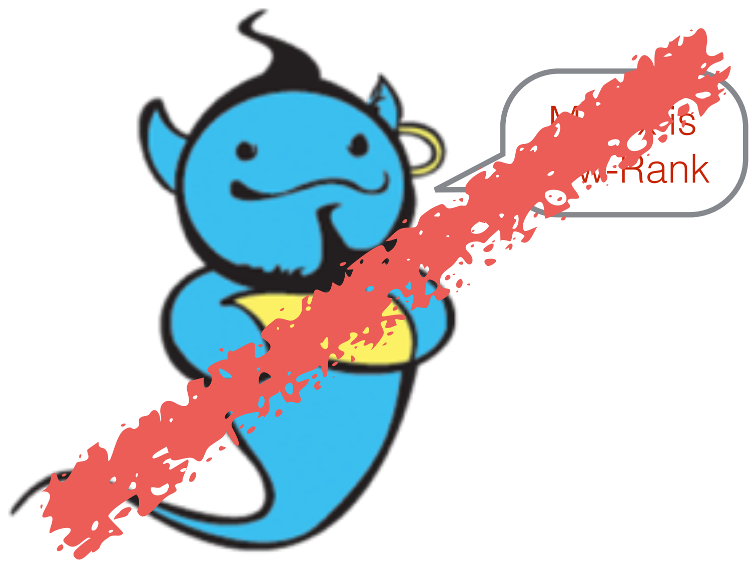
$\Rightarrow$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$



Matrix can be Uniquely Completed

Existing Theory



&

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

- **Conditions on Matrix**
  (e.g., incoherence)
- **Conditions on samples**
  (e.g., uniformly distributed)

$\Rightarrow$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$

Matrix can be Uniquely Completed

Matrix is Low-Rank

$\Leftarrow$

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

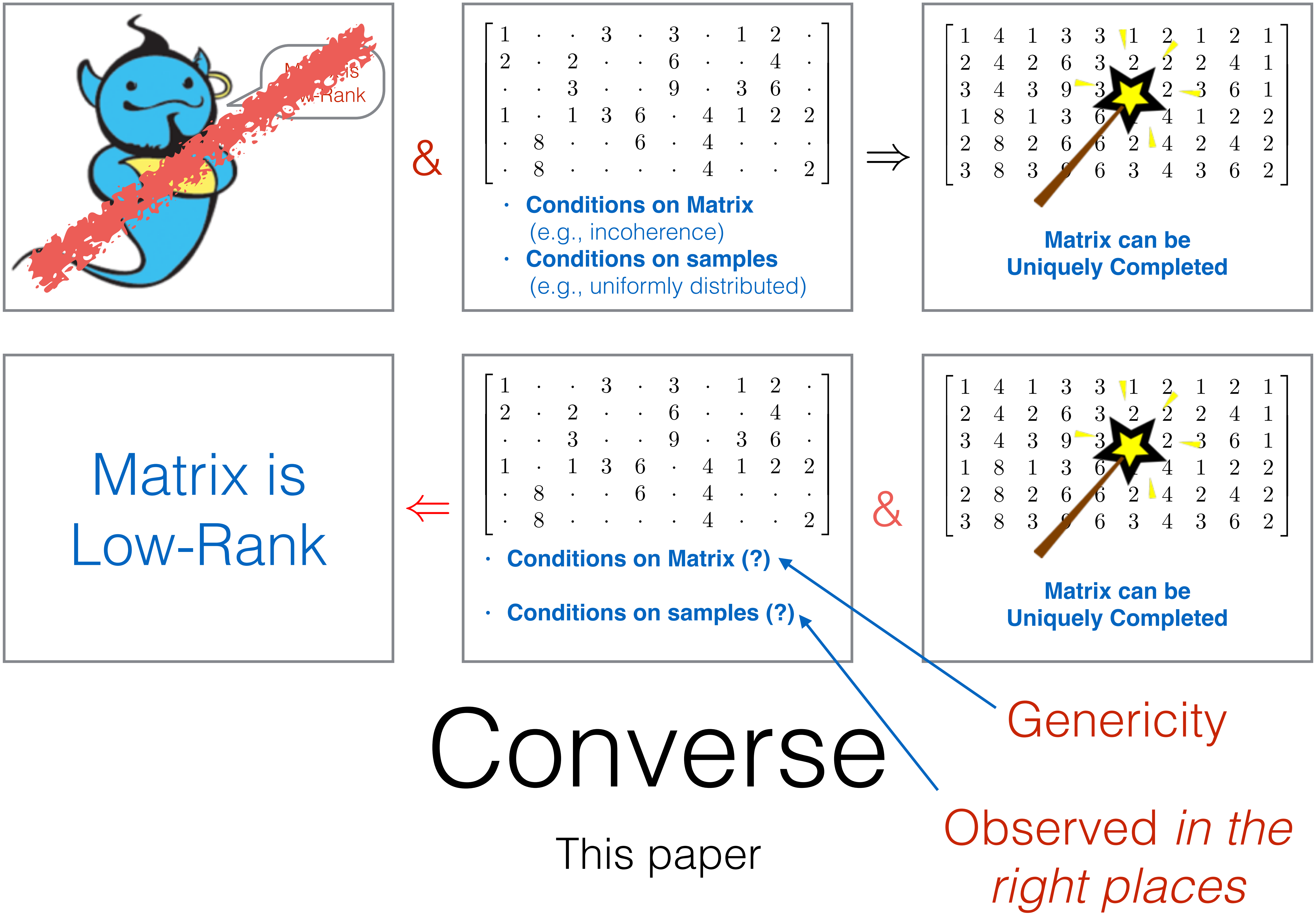
- **Conditions on Matrix (?)**
- **Conditions on samples (?)**

&

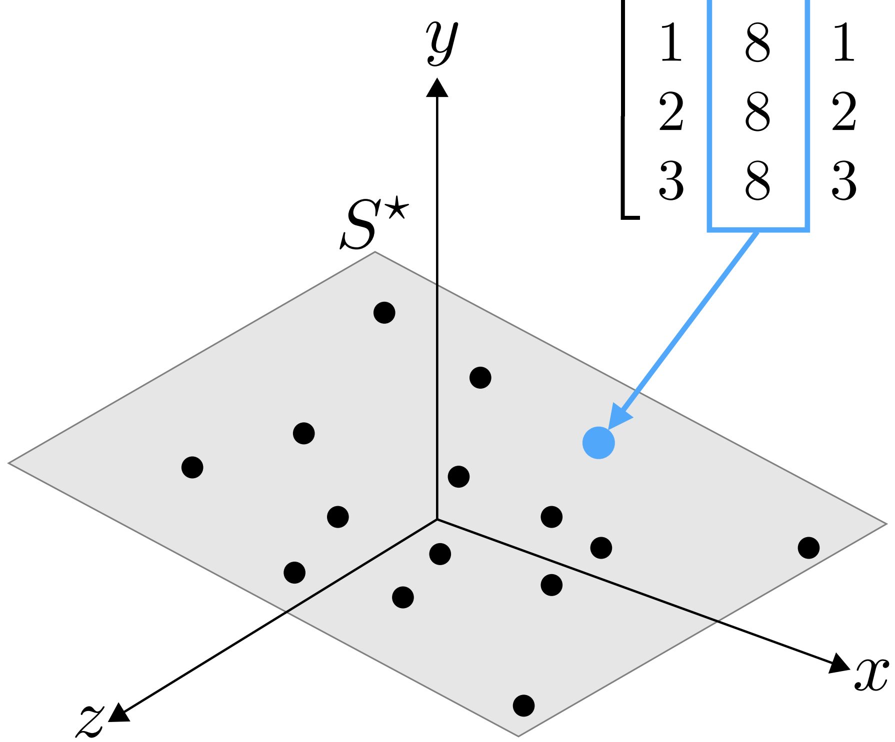
$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$

Matrix can be Uniquely Completed

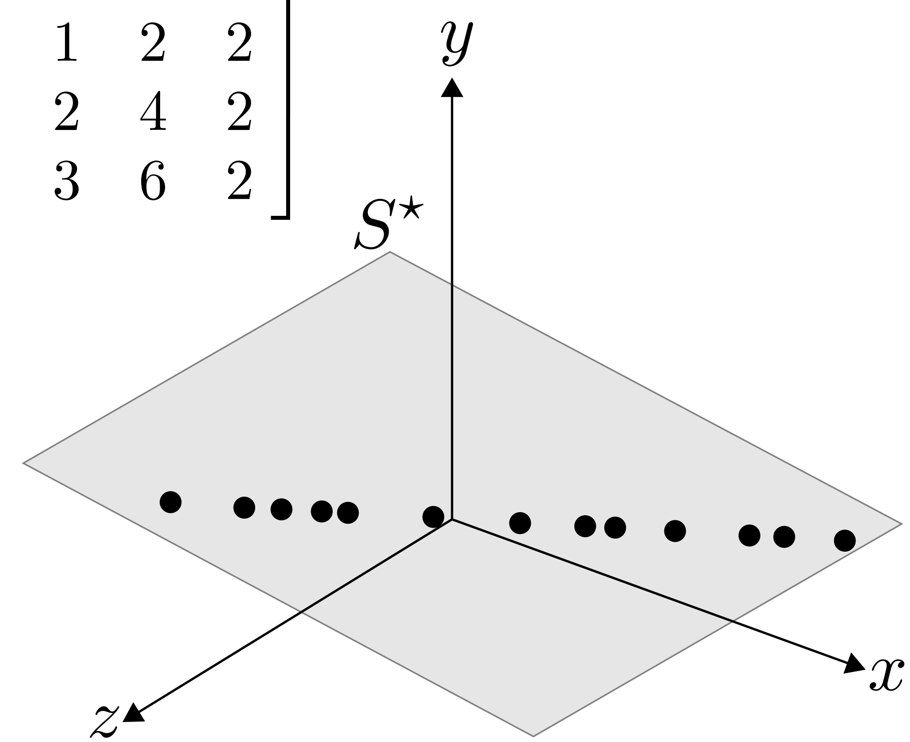
Converse


&
Conditions on Matrix
(e.g., incoherence)
Conditions on samples
(e.g., uniformly distributed)
Matrix can be
Uniquely Completed
Matrix is
Low-Rank
Conditions on Matrix (?)
Conditions on samples (?)
&
Matrix can be
Uniquely Completed
Converse
This paper
Genericity
Observed *in the right places*

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & 3 & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & 1 & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$



Good



Bad

Genericity

What do I mean?

$\mathbf{X}_{\mathbf{\Omega}}$ contains a column, in addition to r disjoint matrices $\{\mathbf{X}_{\mathbf{\Omega}}^{\tau}\}_{\tau=1}^{r}$, each of size $d \times (d-r)$, such that for every τ,

- Every matrix $\mathbf{X}'_{\mathbf{\Omega}}$ formed with a subset of the columns in $\mathbf{X}_{\mathbf{\Omega}}^{\tau}$ satisfies

$$\#rowsWithObservations(\mathbf{X}'_{\mathbf{\Omega}}) \geq \#columns(\mathbf{X}'_{\mathbf{\Omega}}) + r.$$

Observed in the right places

What do I mean?

$\mathbf{X}_{\mathbf{\Omega}}$ contains a column, in addition to r disjoint matrices $\{\mathbf{X}_{\mathbf{\Omega}}^{\tau}\}_{\tau=1}^{r}$, each of size $d \times (d-r)$, such that for every τ,

- Every matrix $\mathbf{X}'_{\mathbf{\Omega}}$ formed with a subset of the columns in $\mathbf{X}_{\mathbf{\Omega}}^{\tau}$ satisfies

$$\#rowsWithObservations(\mathbf{X}'_{\mathbf{\Omega}}) \geq \#columns(\mathbf{X}'_{\mathbf{\Omega}}) + r.$$

$$\mathbf{X}'_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix}$$

$$3 \not\geq 4$$

$$\mathbf{X}'_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & 3 & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix}$$

$$4 \geq 4$$

Observed in the right places

What do I mean?

$\mathbf{X}_\Omega$ contains a column, in addition to r disjoint matrices $\{\mathbf{X}_\Omega^\tau\}_{\tau=1}^r$, each of size $d \times (d-r)$, such that for every τ,

- Every matrix $\mathbf{X}'_\Omega$ formed with a subset of the columns in $\mathbf{X}_\Omega^\tau$ satisfies

$$\#rowsWithObservations(\mathbf{X}'_\Omega) \geq \#columns(\mathbf{X}'_\Omega) + r.$$

$$\mathbf{X}'_\Omega = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix} \qquad \mathbf{X}'_\Omega = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & 3 & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix}$$

$$3 \not\geq 4 \qquad\qquad 4 \geq 4$$

Observed in the right places

What do I mean?

(These conditions are met with high probability)

Suppose $\mathbf{X}_{\mathbf{\Omega}}$ is a *generic* matrix *observed in the right places*. If there is a rank-r matrix that agrees with $\mathbf{X}$ on $\mathbf{\Omega}$, then $\mathbf{X}$ is indeed rank-r with probability 1.

Main Result

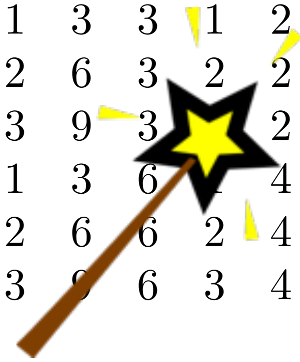
Suppose $\mathbf{X_\Omega}$ is a *generic* matrix *observed in the right places*. If there is a rank-r matrix that agrees with $\mathbf{X}$ on $\mathbf{\Omega}$, then $\mathbf{X}$ is indeed rank-r with probability 1.

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

- **Conditions on Matrix**
  (genericity)
- **Conditions on samples**
  (observed in the right places)

&

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & \text{[illegible]} & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & \text{[illegible]} & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & \text{[illegible]} & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$



Matrix can be Completed

$\Rightarrow$

Matrix is Low-Rank

Main Result

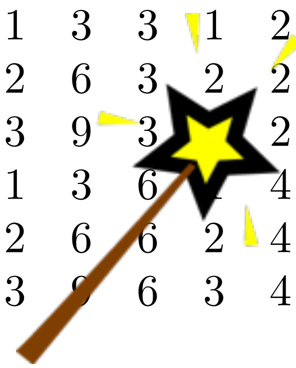
This is just a first step

$$\begin{bmatrix} 1 & \cdot & \cdot & 3 & \cdot & 3 & \cdot & 1 & 2 & \cdot \\ 2 & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & 9 & \cdot & 3 & 6 & \cdot \\ 1 & \cdot & 1 & 3 & 6 & \cdot & 4 & 1 & 2 & 2 \\ \cdot & 8 & \cdot & \cdot & 6 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 8 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 2 \end{bmatrix}$$

- **Conditions on Matrix** (genericity)
- **Conditions on samples** (observed in the right places)

&

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 6 & 3 & 2 & 2 & 2 & 4 & 1 \\ 3 & 4 & 3 & 9 & 3 & & 2 & 3 & 6 & 1 \\ 1 & 8 & 1 & 3 & 6 & & 4 & 1 & 2 & 2 \\ 2 & 8 & 2 & 6 & 6 & 2 & 4 & 2 & 4 & 2 \\ 3 & 8 & 3 & 9 & 6 & 3 & 4 & 3 & 6 & 2 \end{bmatrix}$$



Matrix can be Approximated by Low-Rank

$\Rightarrow$

Matrix is Approximately Low-Rank

What we would like to say

(Future work)

Gràcies