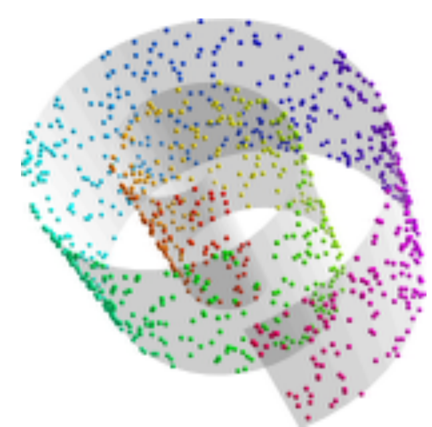
Low Algebraic Dimension Matrix Completion



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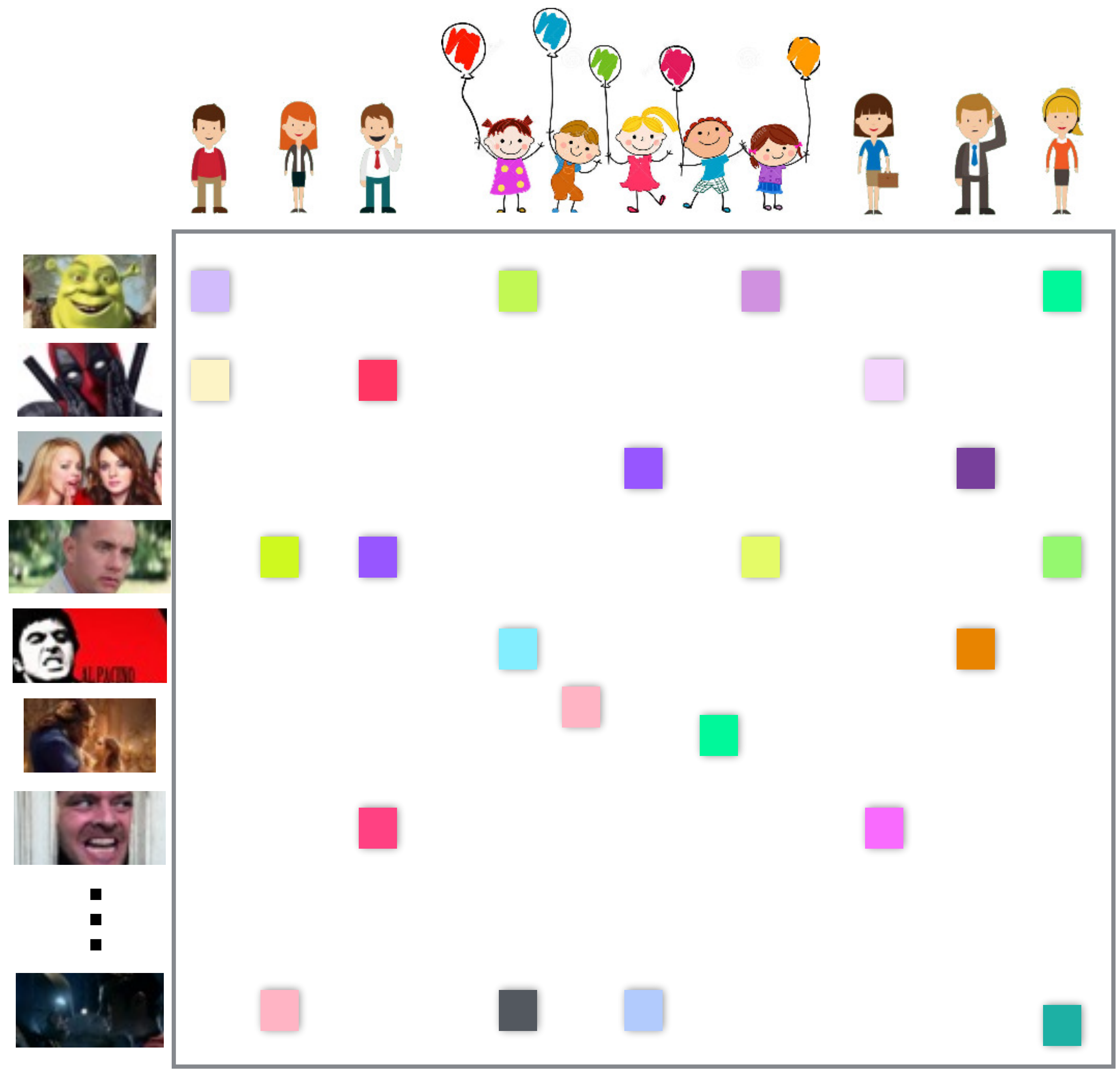
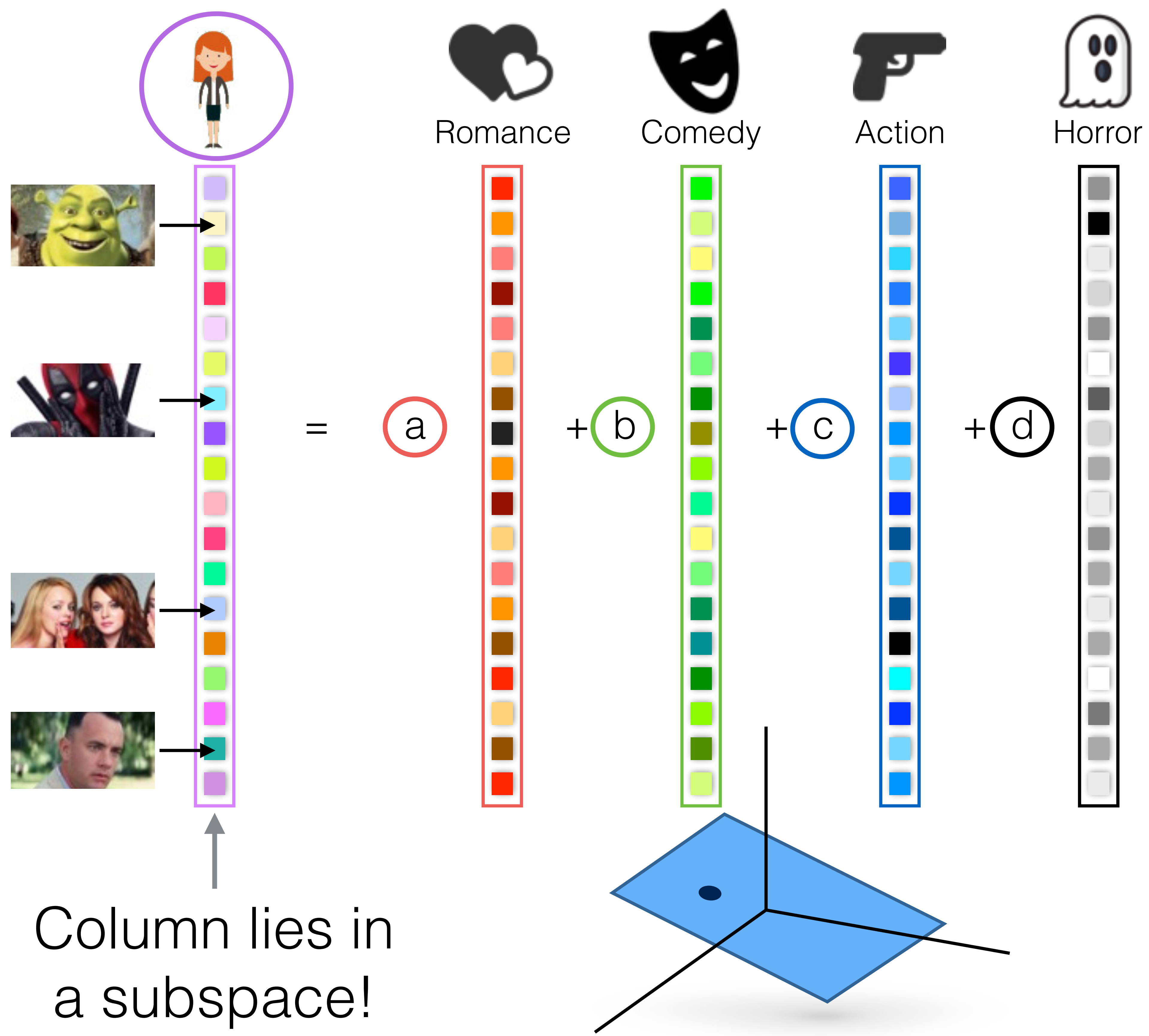
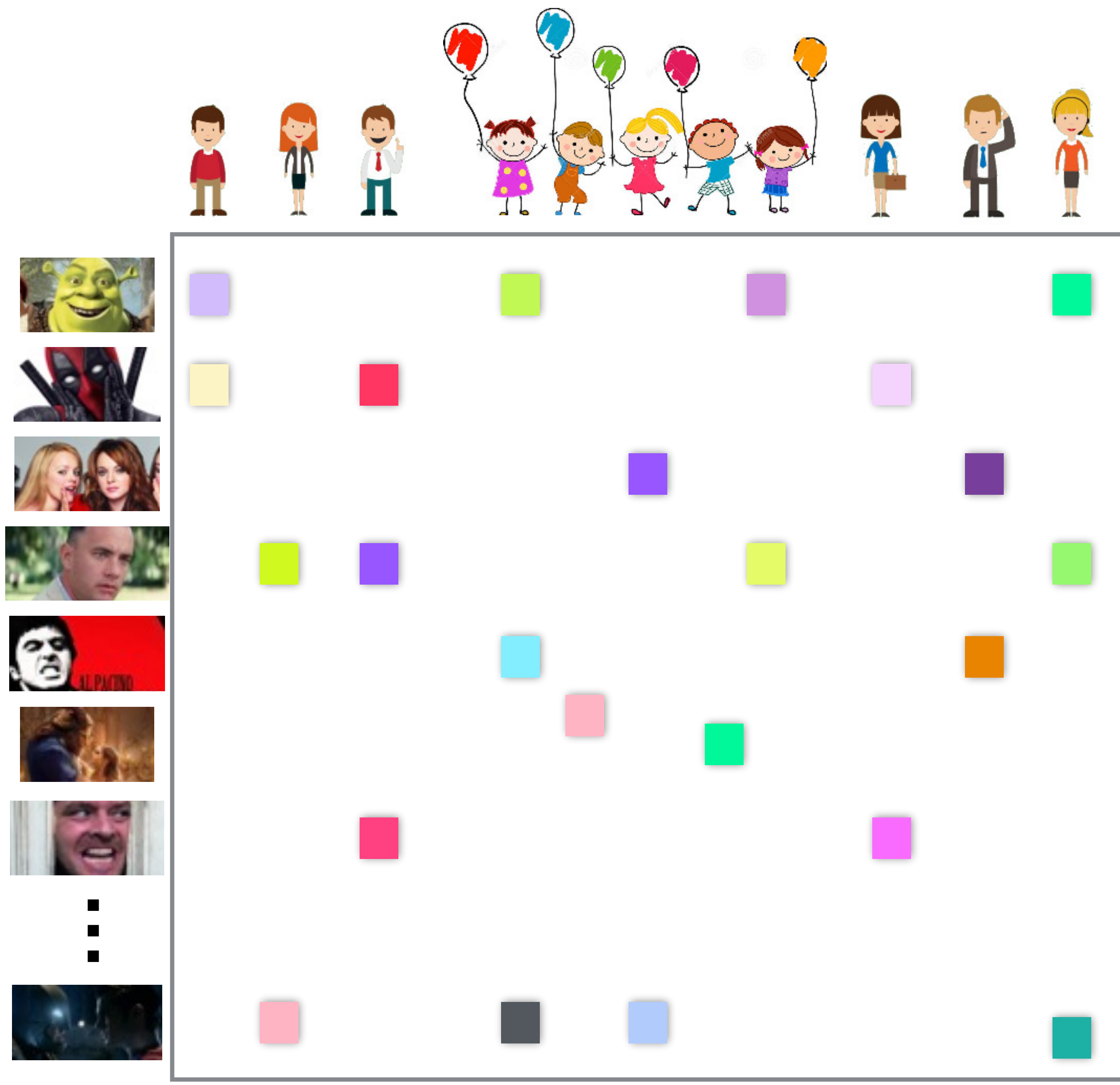


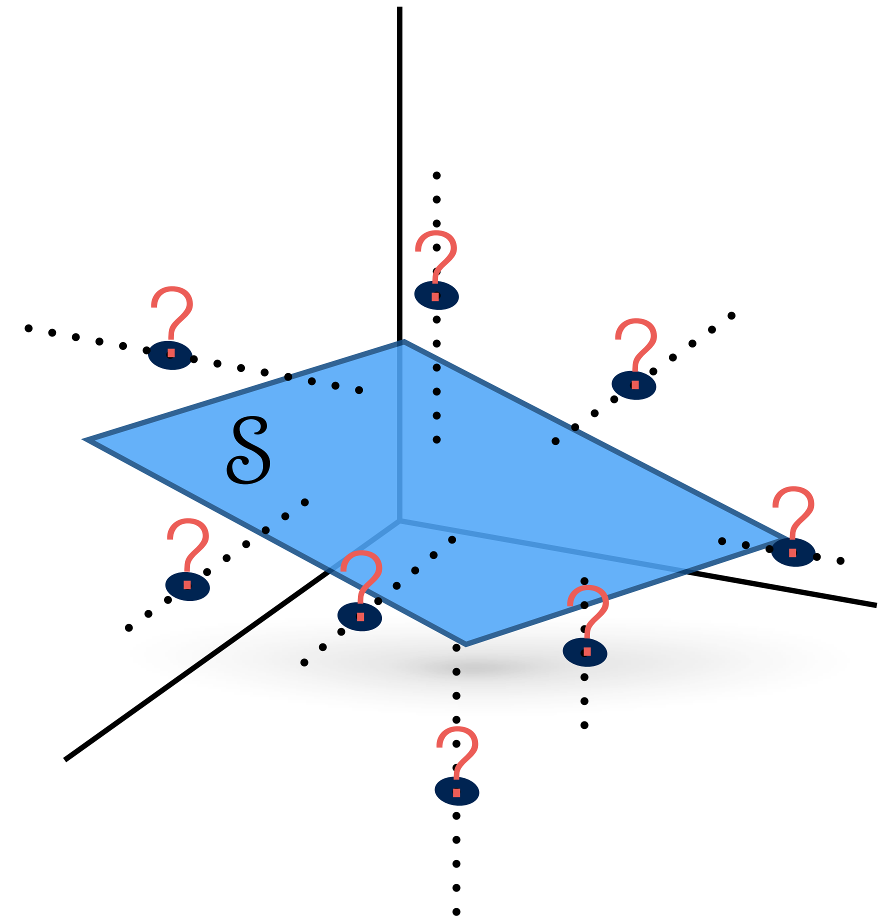
Textbook Example

Recommender Systems



Romance
Comedy
Action
Horror
=
a
+ b
+ c
+ d
Column lies in
a subspace!

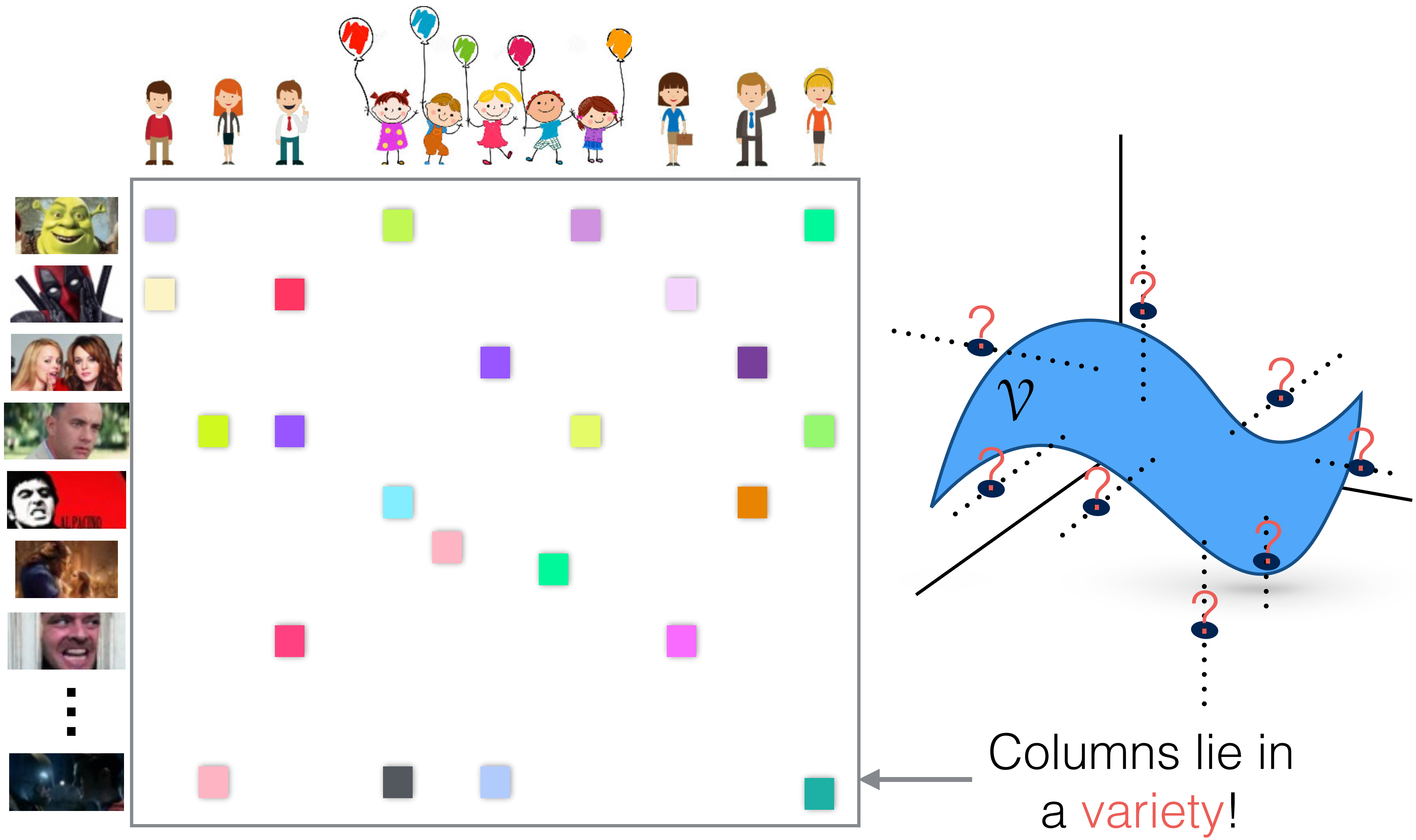




Columns lie in a subspace!

We want to find this Subspace!

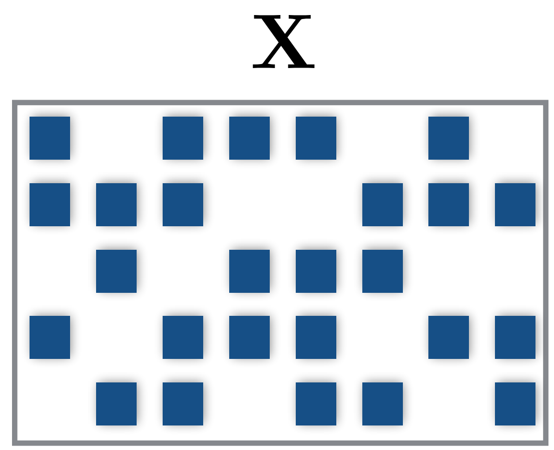
Problem is: data is incomplete!



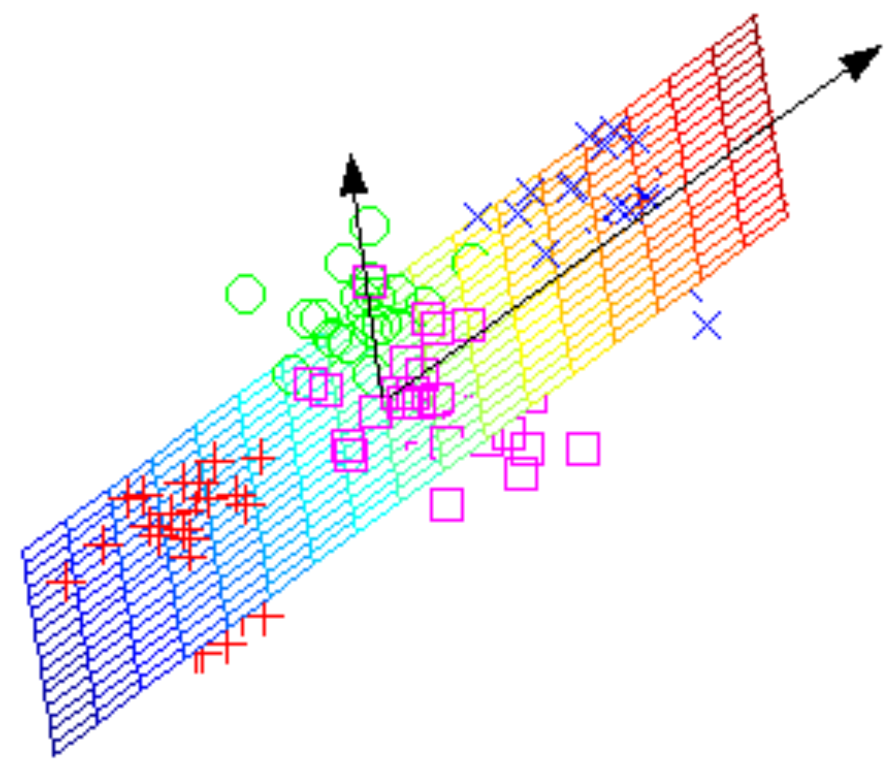
What if dependencies are nonlinear?!

Problem is: data is incomplete!

What am I telling you?



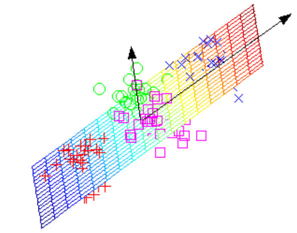
$\in$



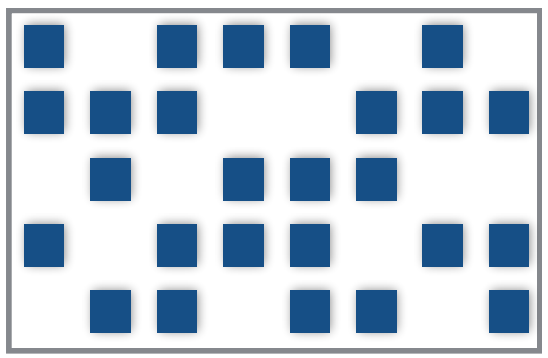
Given: Incomplete
data matrix

Goal: Find linear subspace
that explains data

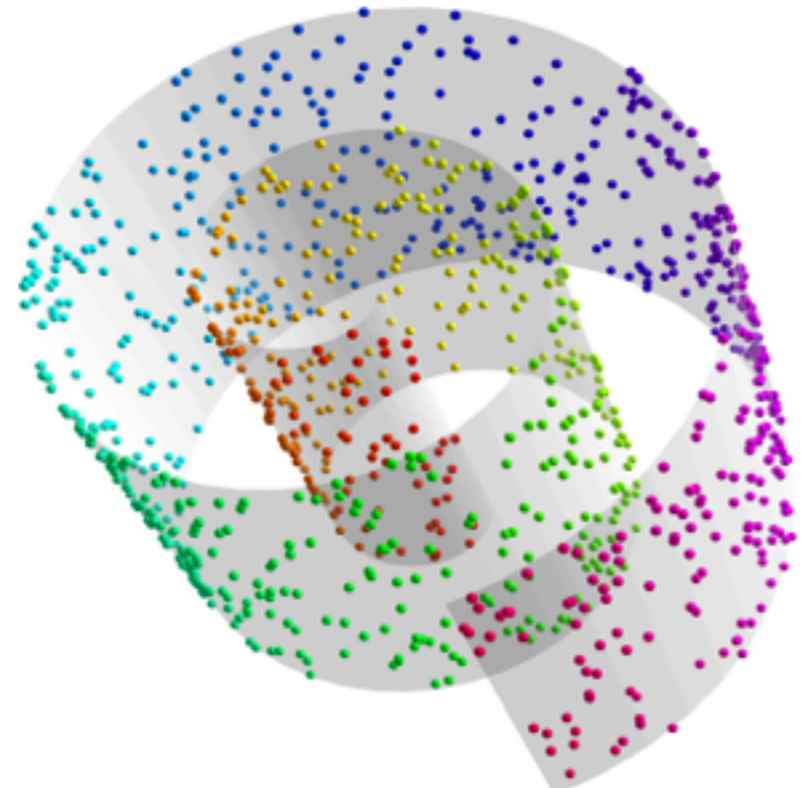
Low-Rank Matrix Completion



$\mathbf{X}$



$\in$



Given: Incomplete data matrix

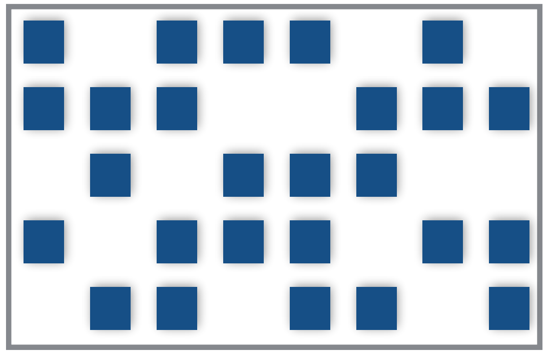
Algebraic Variety

Goal: Find ~~linear subspace~~ that explains data

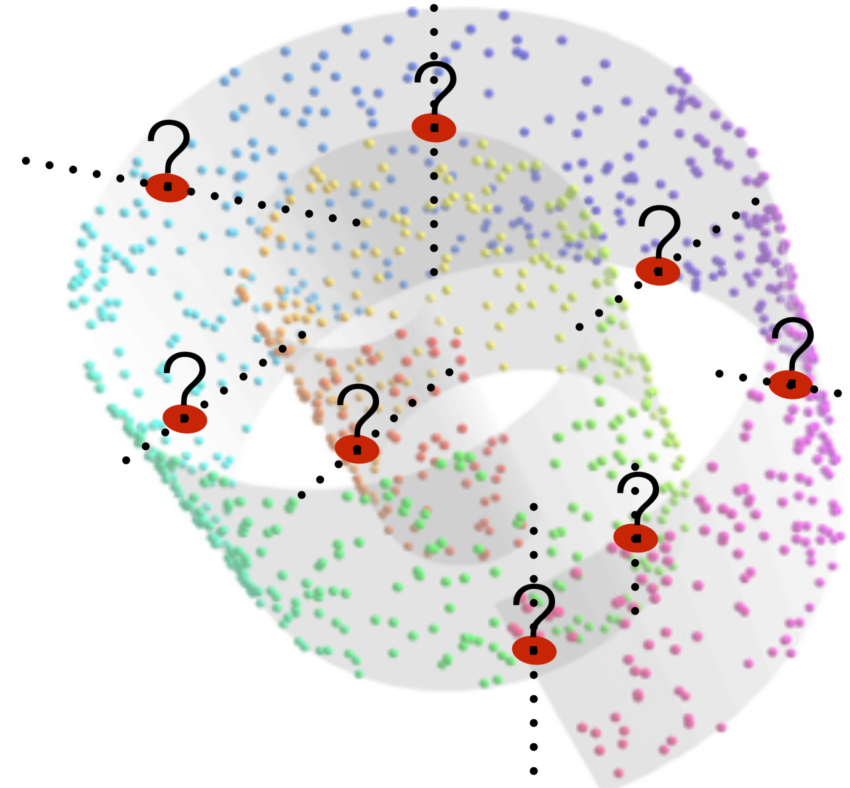
~~Low-Rank~~ Matrix Completion

Low Algebraic Dimension

$\mathbf{X}$



$\in$



Given: Incomplete data matrix

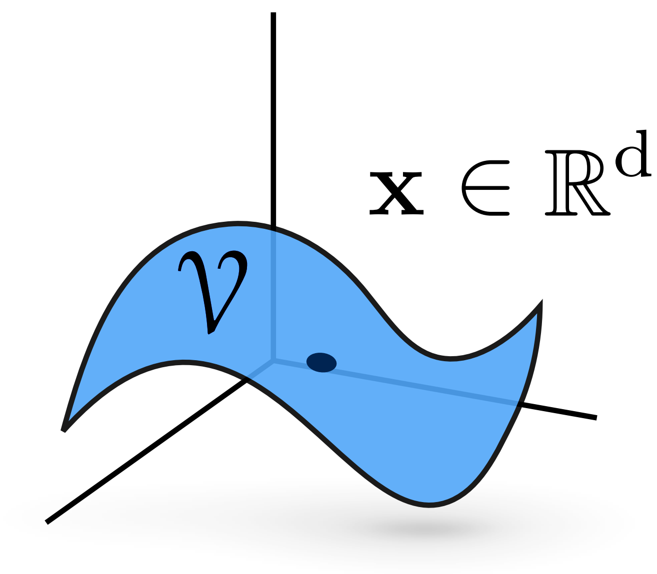
Goal: Find ~~linear subspace~~ Algebraic Variety that explains data

Is it possible? Yes

When? When you observe *the right entries*

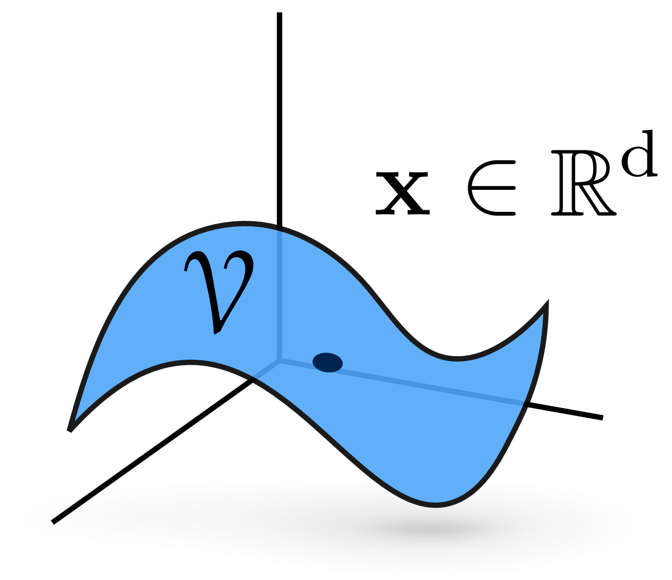
How? Using tensors

ROLL UP YOUR SLEEVES!



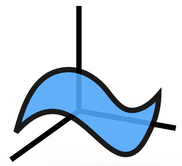
$$\begin{cases} f_1(\mathbf{x}) = 0 \\ f_2(\mathbf{x}) = 0 \\ \vdots \\ f_{\mathrm{N}}(\mathbf{x}) = 0 \end{cases}$$

Consider a point in a variety



$$\begin{cases} v_{11}x_1^2 + v_{12}x_1x_2 + v_{13}x_1x_3 + \cdots + v_{1D}x_d^2 = 0 \\ v_{21}x_1^2 + v_{22}x_1x_2 + v_{23}x_1x_3 + \cdots + v_{2D}x_d^2 = 0 \\ \vdots \\ v_{N1}x_1^2 + v_{N2}x_1x_2 + v_{N3}x_1x_3 + \cdots + v_{ND}x_d^2 = 0 \end{cases}$$

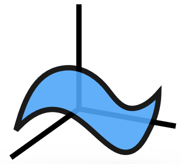
Consider a point in a variety



$$\mathbf{x} \in \mathcal{V}$$

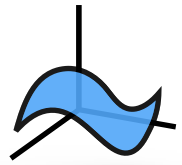
$$v_{11}x_1^2 + v_{12}x_1x_2 + v_{13}x_1x_3 + \cdots + v_{1D}x_d^2 = 0$$

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1D} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ \vdots \\ x_d^2 \end{bmatrix} = 0$$



$$\mathbf{x} \in \mathcal{V} \qquad \begin{cases} v_{11}x_1^2 + v_{12}x_1x_2 + v_{13}x_1x_3 + \cdots + v_{1D}x_d^2 &= 0 \\ v_{21}x_1^2 + v_{22}x_1x_2 + v_{23}x_1x_3 + \cdots + v_{2D}x_d^2 &= 0 \end{cases}$$

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1D} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2D} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ \vdots \\ x_d^2 \end{bmatrix} = \mathbf{0}$$



$$\mathbf{x} \in \mathcal{V} \begin{cases} v_{11}x_1^2 + v_{12}x_1x_2 + v_{13}x_1x_3 + \cdots + v_{1D}x_d^2 = 0 \\ v_{21}x_1^2 + v_{22}x_1x_2 + v_{23}x_1x_3 + \cdots + v_{2D}x_d^2 = 0 \\ \vdots \\ v_{N1}x_1^2 + v_{N2}x_1x_2 + v_{N3}x_1x_3 + \cdots + v_{ND}x_d^2 = 0 \end{cases}$$

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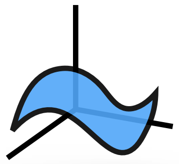
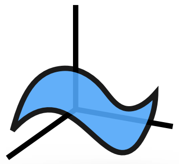


$$\mathbf{x} \in \mathcal{V} \begin{cases} v_{11}x_1^2 + v_{12}x_1x_2 + v_{13}x_1x_3 + \cdots + v_{1D}x_d^2 = 0 \\ v_{21}x_1^2 + v_{22}x_1x_2 + v_{23}x_1x_3 + \cdots + v_{2D}x_d^2 = 0 \\ \vdots \\ v_{N1}x_1^2 + v_{N2}x_1x_2 + v_{N3}x_1x_3 + \cdots + v_{ND}x_d^2 = 0 \end{cases}$$

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1D} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2D} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & v_{N3} & \cdots & v_{ND} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ \vdots \\ x_d^2 \end{bmatrix} = \mathbf{0}$$

$$V^{\mathsf{T}} \mathbf{x}^{\otimes \mathbf{2}} = \mathbf{0}$$



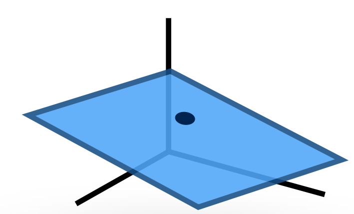
$$\mathbf{x} \in \mathcal{V} \iff V^{\mathsf{T}} \mathbf{x}^{\otimes \mathbf{2}} = \mathbf{0}$$

$$\Updownarrow$$

$$\mathbf{x}^{\otimes \mathbf{2}} \in \ker V^{\mathsf{T}}$$

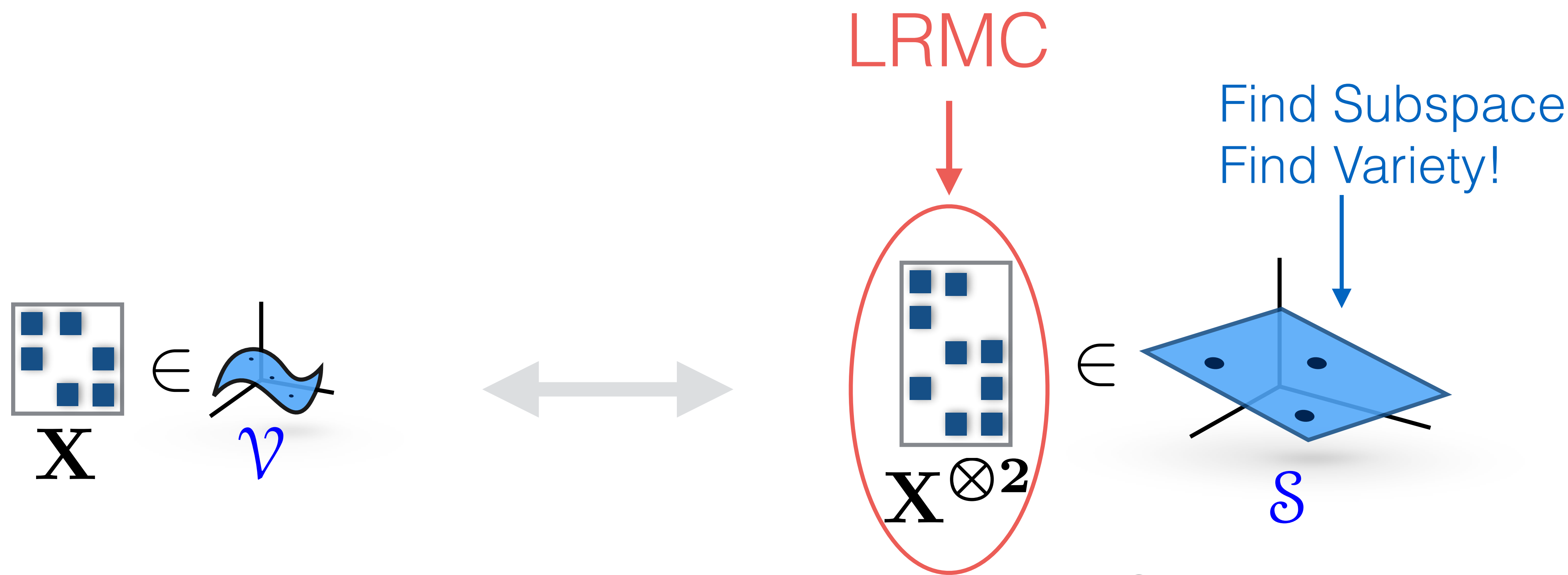
$$\|$$

$$\mathbf{x}^{\otimes \mathbf{2}} \in \mathcal{S}$$



Variety = Tensorized Subspace!

What does this mean?



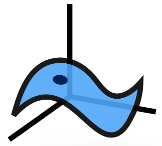
Algebraic Variety
(nonlinear)

Tensorized Subspace
(linear)

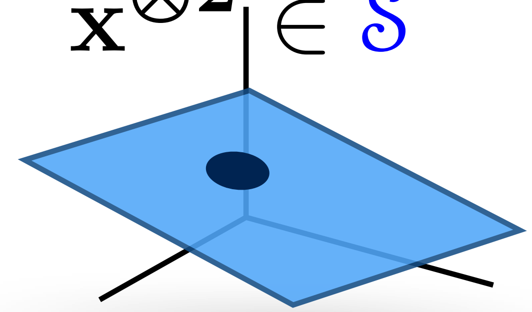
What does this mean?

Is this just standard Low-Rank Matrix Completion?
More or less…

$\mathbf{x} \in \mathcal{V}$

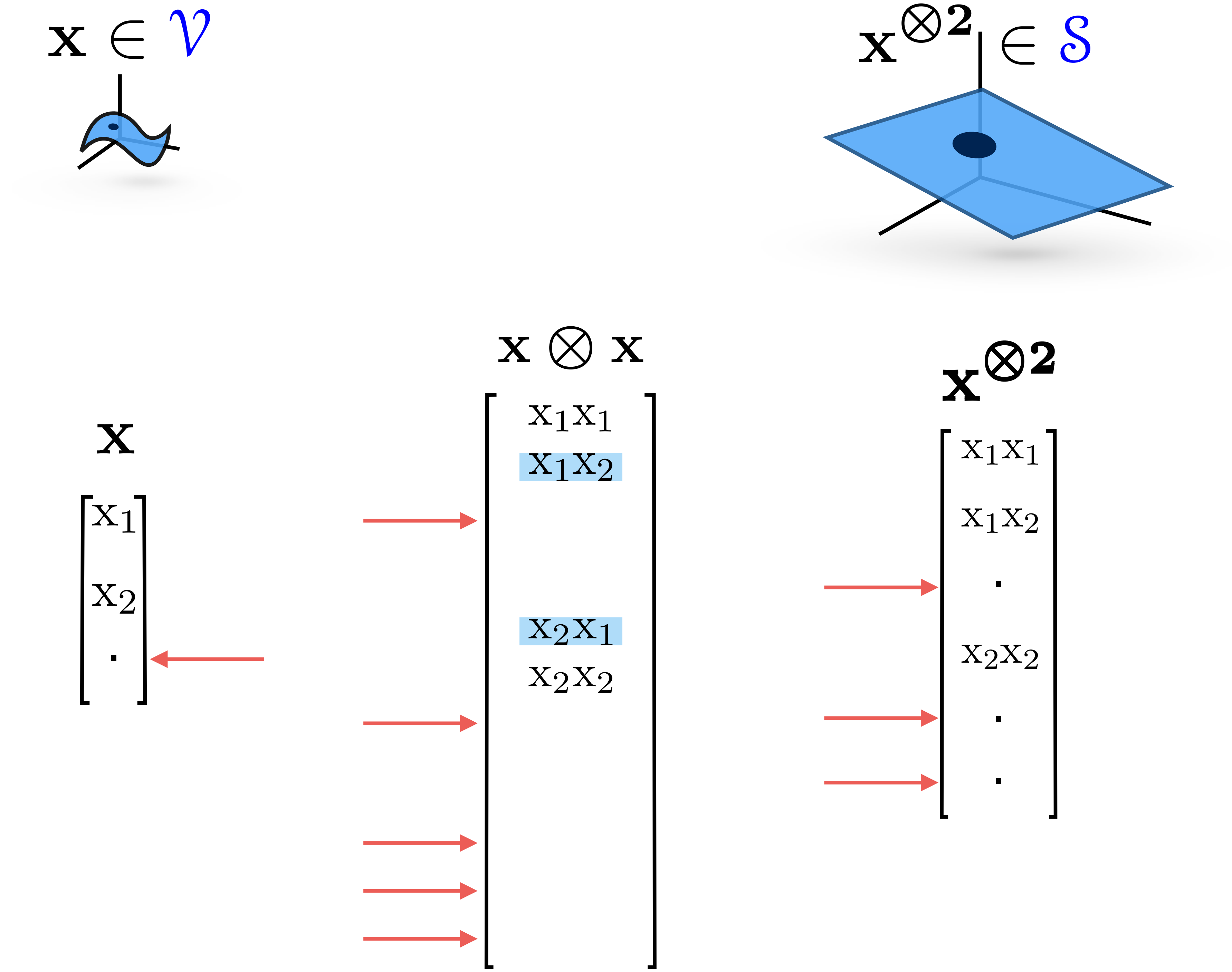


$\mathbf{x}^{\otimes 2} \in \mathcal{S}$



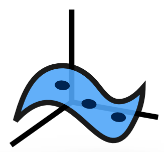
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \mathbf{x} \otimes \mathbf{x} = \begin{bmatrix} x_1x_1 \\ x_1x_2 \\ x_1x_3 \\ x_2x_1 \\ x_2x_2 \\ x_2x_3 \\ x_3x_1 \\ x_3x_2 \\ x_3x_3 \end{bmatrix} \rightarrow \mathbf{x}^{\otimes 2} = \begin{bmatrix} x_1x_1 \\ x_1x_2 \\ x_1x_3 \\ x_2x_2 \\ x_2x_3 \\ x_3x_3 \end{bmatrix}$$

Recall…

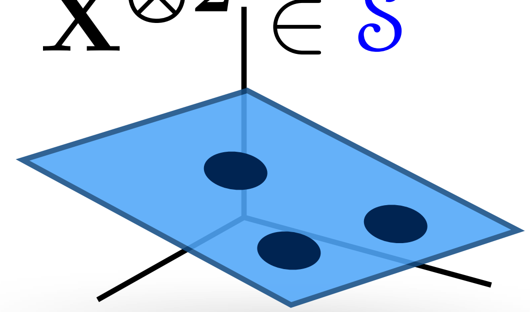


The sampling is highly restricted!

$\mathbf{X} \in \mathcal{V}$



$\mathbf{X}^{\otimes 2} \in \mathcal{S}$



$$\mathbf{X} = \begin{bmatrix} x_1 & x_1 & \cdot \\ x_2 & \cdot & x_2 \\ \cdot & x_3 & x_3 \end{bmatrix}$$

$$\mathbf{X}^{\otimes 2} = \underbrace{\begin{bmatrix} x_1^2 & x_1^2 & \cdot \\ x_1x_2 & \cdot & \cdot \\ \cdot & x_1x_3 & \cdot \\ x_2^2 & \cdot & x_2^2 \\ \cdot & \cdot & x_2x_3 \\ \cdot & x_3^2 & x_3^2 \end{bmatrix}}_{3/20}$$

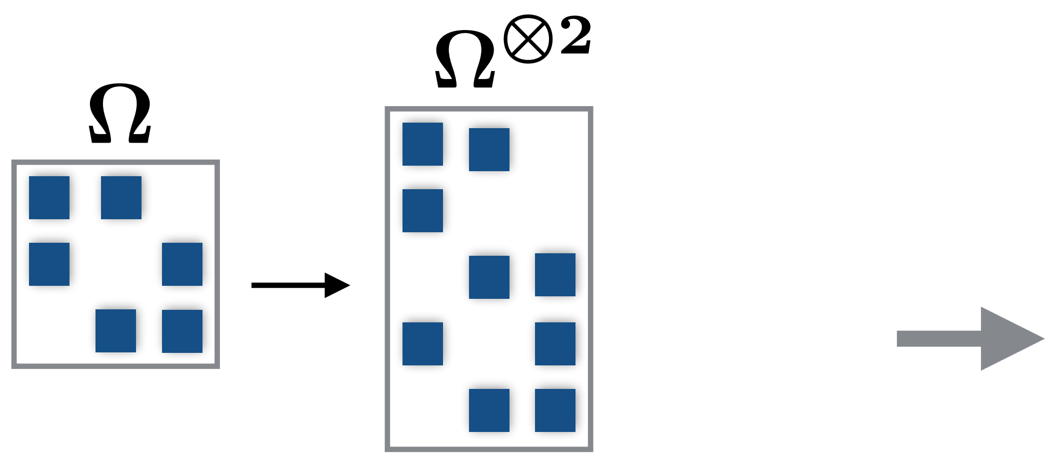
Impossible!

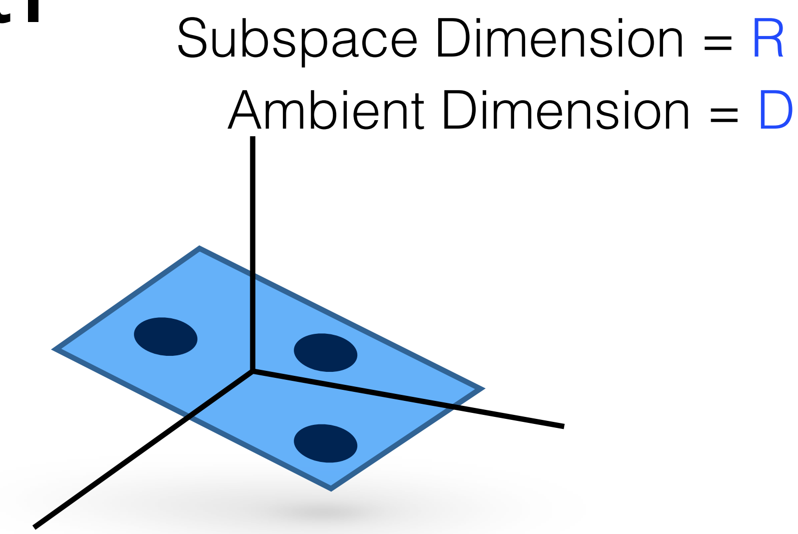
$$\underbrace{\begin{bmatrix} x_1^2 & x_1^2 & x_1^2 & & \cdot \\ \cdot & x_1x_2 & x_1x_2 & & \cdot \\ \cdot & x_1x_3 & \cdot & & \cdot \\ x_2^2 & \cdot & \cdot & \cdots & x_2^2 \\ \cdot & \cdot & x_2x_3 & & x_2x_3 \\ x_3^2 & \cdot & \cdot & & x_3^2 \end{bmatrix}}_{17/20}$$

Small letters in LRMC:

Incoherence and Uniform Sampling

In general





Given: available samplings

Can we find $\mathcal{S}$?

Theorem (P.-A., Ongie, Balzano, Willett, Nowak, 2017)

Suppose $\mathcal{V}$ is in general position. With probability 1, $\mathcal{S}$ can be uniquely recovered if and only if there is a matrix $\mathbf{\Omega}_\star^{\otimes 2}$ formed with D-R columns of $\mathbf{\Omega}^{\otimes 2}$ such that every $\mathbf{\Omega}_\ell^{\otimes 2}$ formed with a subset of columns in $\mathbf{\Omega}_\star^{\otimes 2}$ satisfies:

$$\#\text{rows\_with\_observations}(\mathbf{\Omega}_\ell^{\otimes 2}) \geq \#\text{columns}(\mathbf{\Omega}_\ell^{\otimes 2}) + \text{R}.$$

Furthermore, this condition is true if and only if $\dim \ker \mathbf{A}^\mathsf{T} = \text{R}$, whence $\mathcal{S} = \ker \mathbf{A}^\mathsf{T}$.

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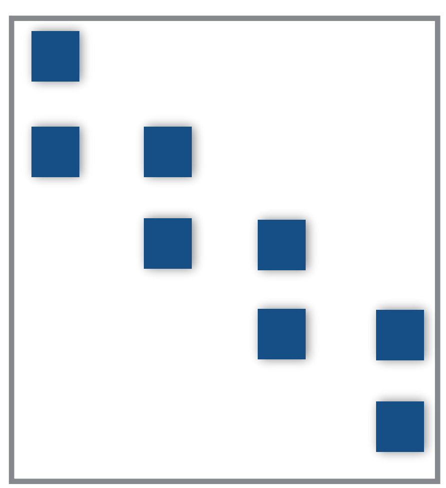
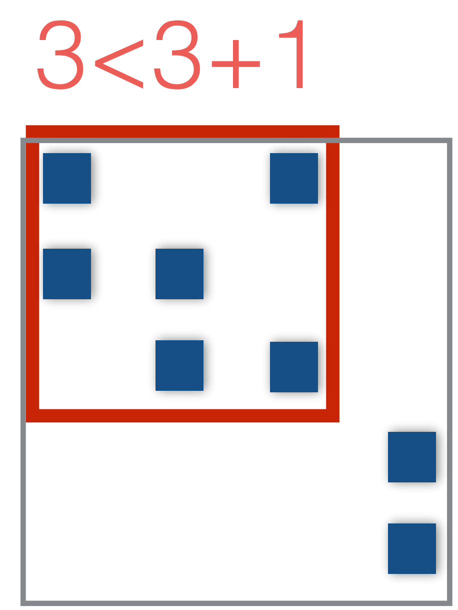
Furthermore, this condition is true if and only if $\dim \ker \mathbf{A}^\mathsf{T} = \text{R}$, whence $\mathcal{S} = \ker \mathbf{A}^\mathsf{T}$.

D=5, R=1

$\mathbf{\Omega}^{\otimes 2} =$ 

Good

$\mathbf{\Omega}^{\otimes 2} =$ 

Bad

Theorem (P.-A., Ongie, Balzano, Willett, Nowak, 2017)

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Furthermore, this condition is true if and only if $\dim \ker \mathbf{A}^\mathsf{T} = \text{R}$, whence $\mathcal{S} = \ker \mathbf{A}^\mathsf{T}$.

In words:

- Yes, it is possible to find the subspace $\mathcal{S}$.
- Iff you observe *the right* entries (rows vs cols condition).
- There is an easy way to check this rows vs cols condition.
- If the condition is satisfied, there is an easy way to find $\mathcal{S}$.

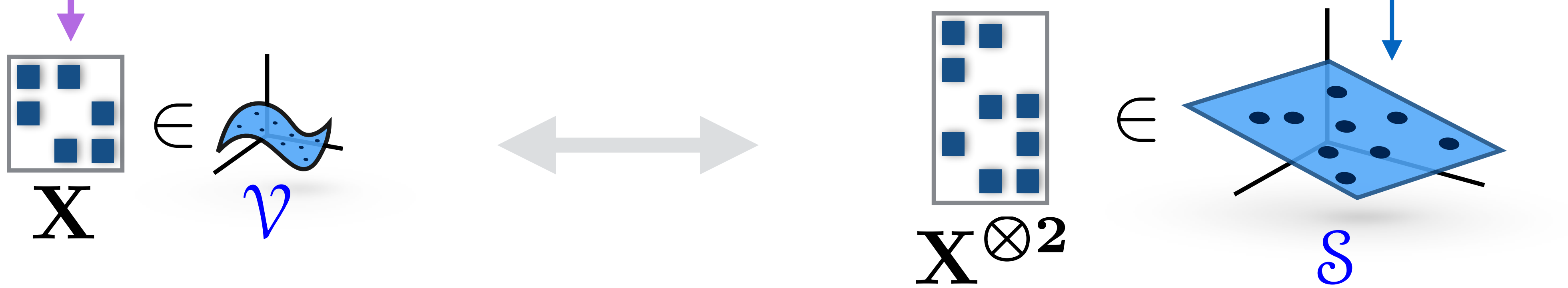
So, what do we know so far?

Is it even possible that these produce *the right entries*?

Yes :)

We need to observe *the right entries*

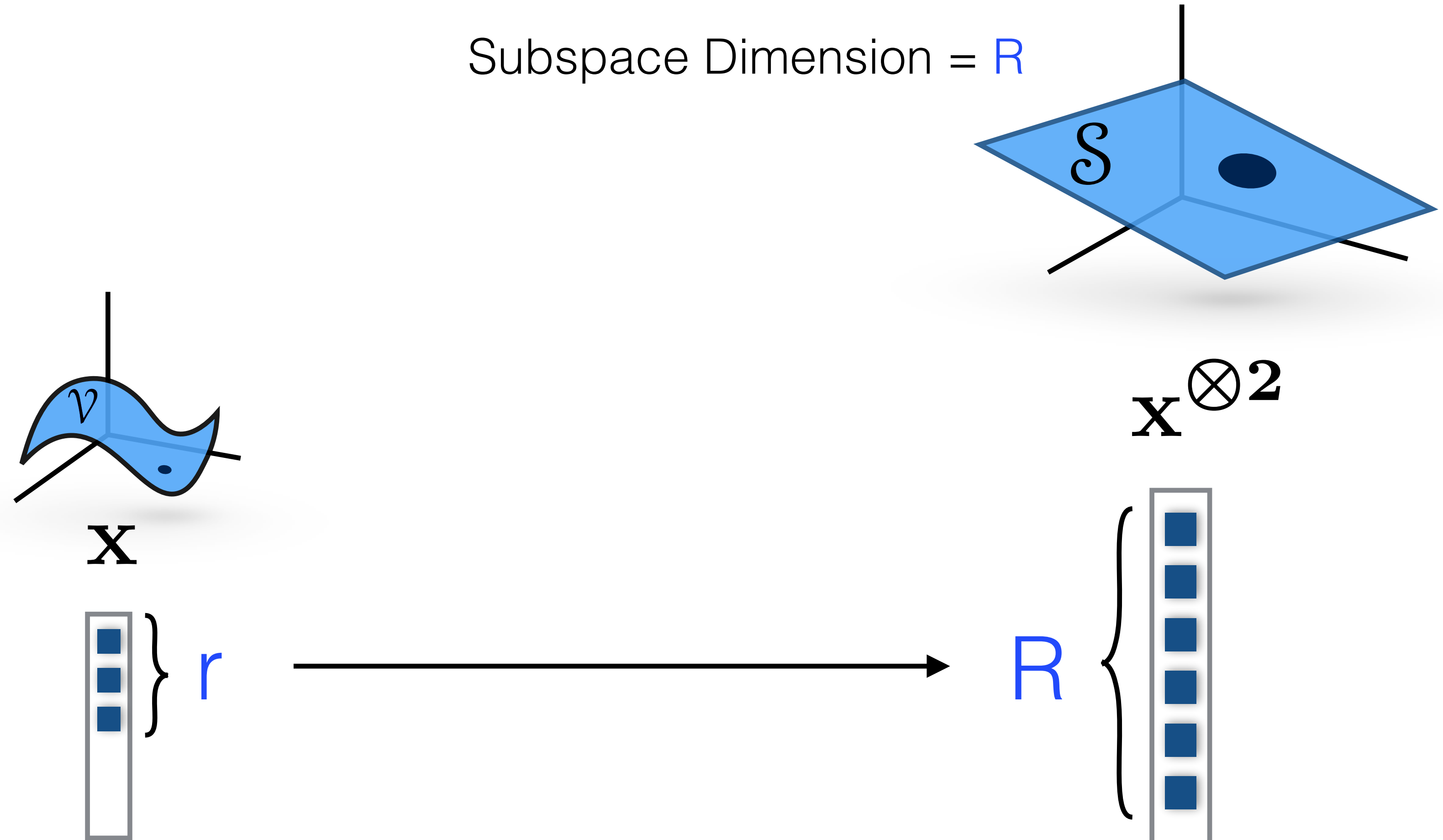
Find Subspace, Find Variety!



Algebraic Variety (nonlinear)

Tensorized Subspace (linear)

So, what do we know so far?



Theorem (P.-A., Ongie, Balzano, Willett, Nowak, 2017)

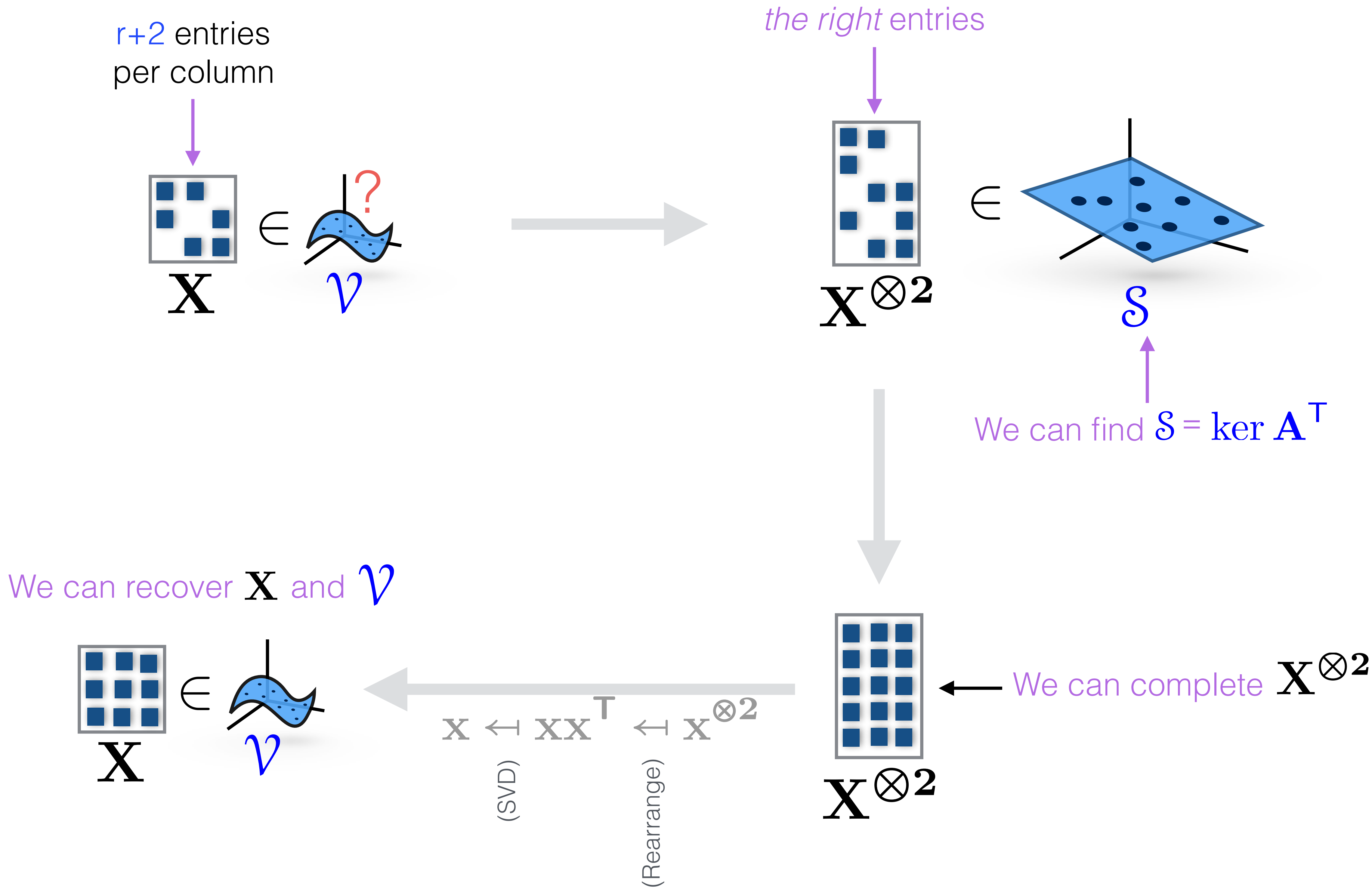
Suppose $\mathcal{V}$ is in general position. Suppose each column $\mathbf{x}$ has m samples.

(i) If $m < r$, then $\mathcal{S}$ cannot be uniquely determined.

(ii) There are cases with $m = r$ and $m = r+1$ where $\mathcal{S}$ cannot be uniquely determined.

(iii) If $m \geq r+2$, then $\mathcal{S}$ can be uniquely determined (if you observe *the right* entries).

So, what do we know so far?



So, what do we know so far?

(Provable Algorithm)

Thank you!

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