Mixture Matrix Completion:

Theory, Algorithms and Open Questions

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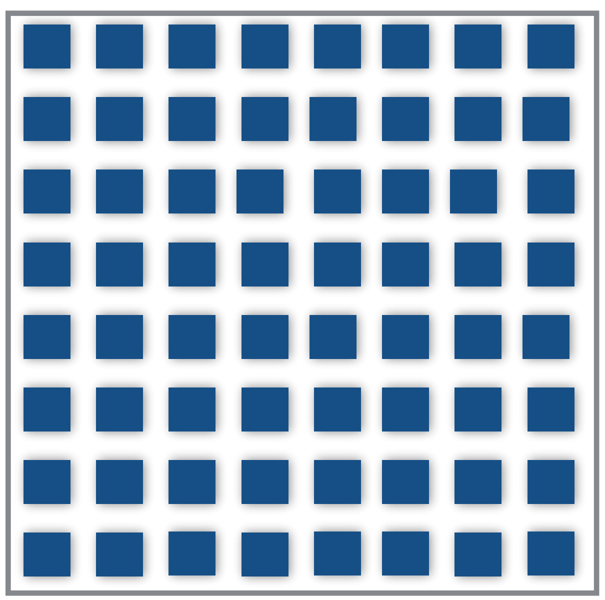
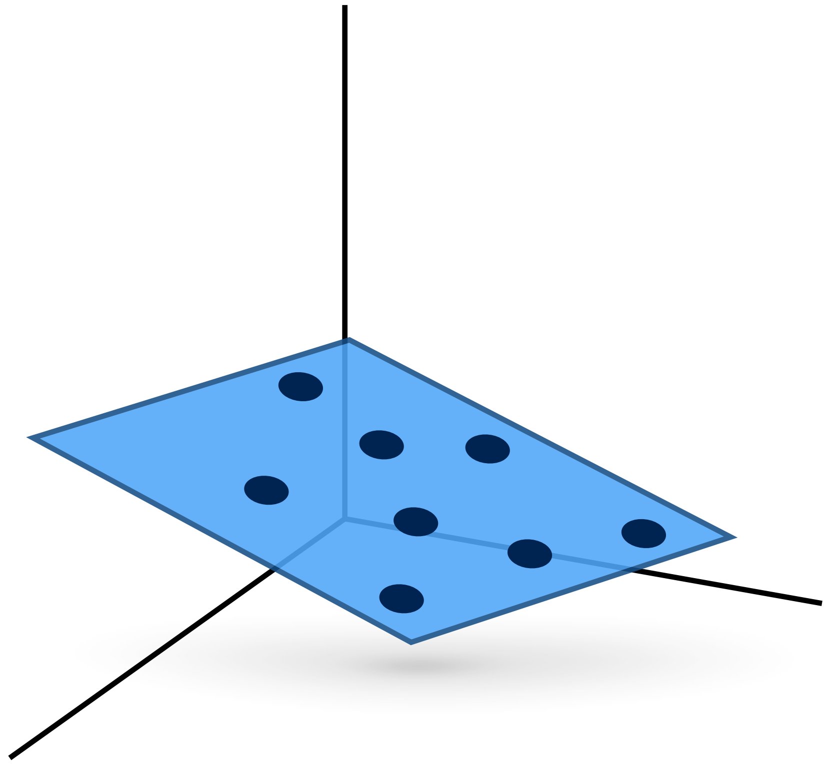
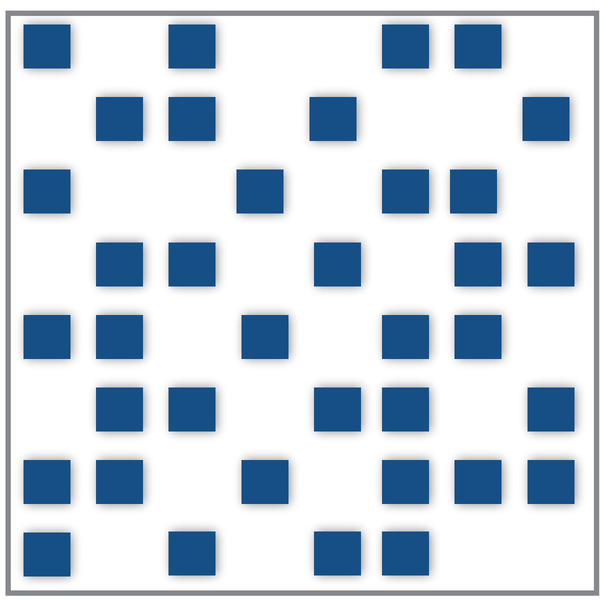
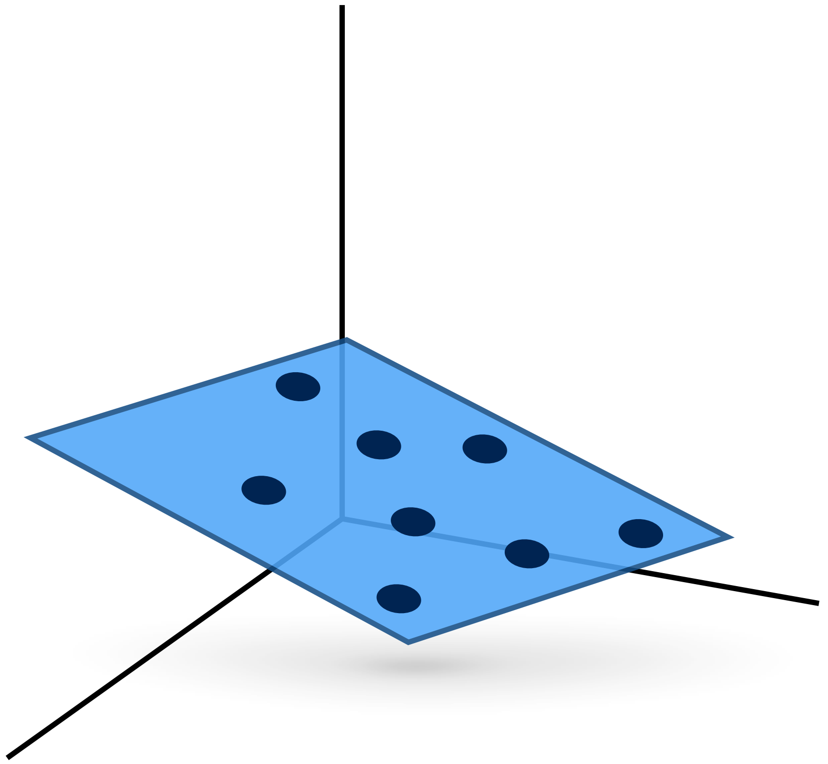
SIAM - Optimization, 2017





Low-Rank Matrix Completion





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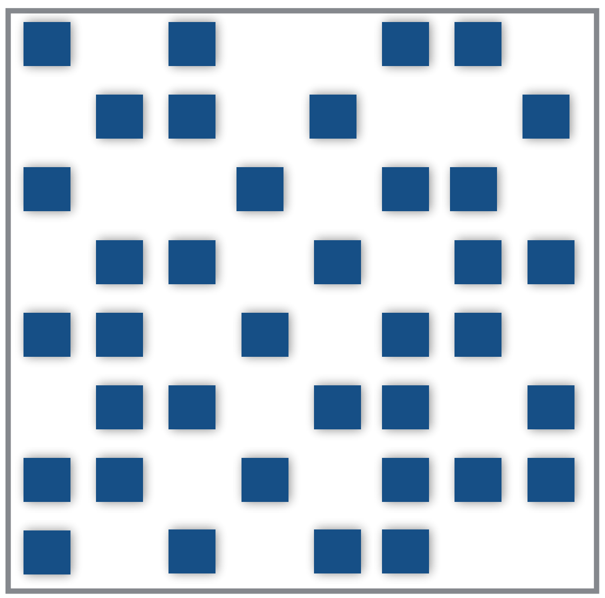

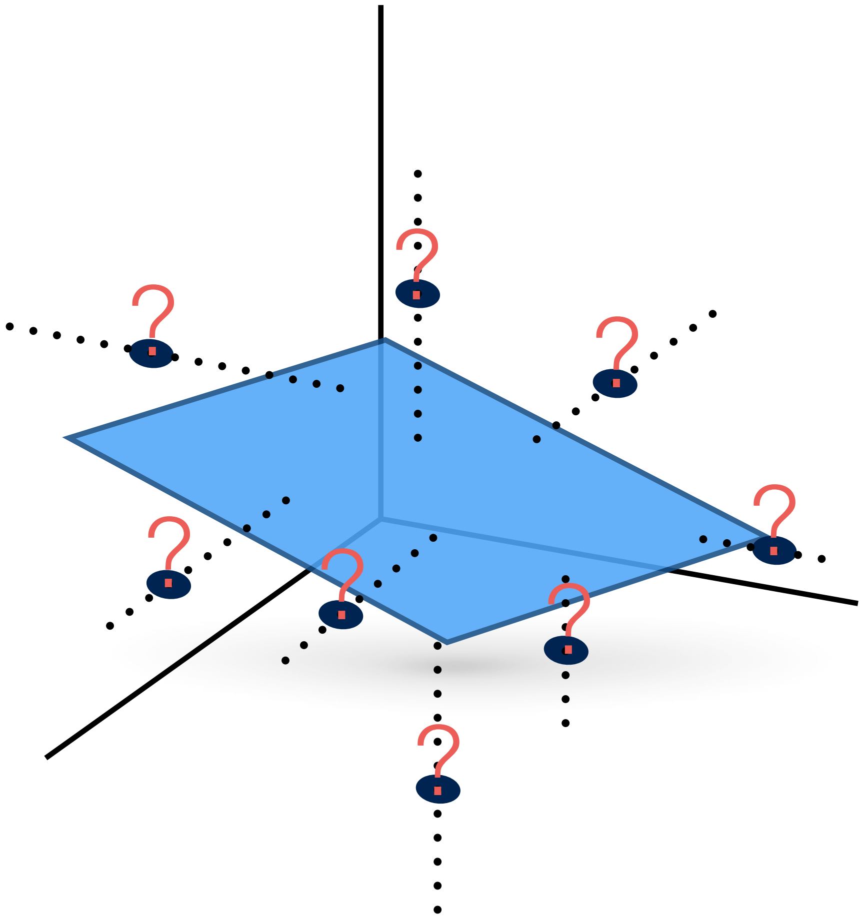

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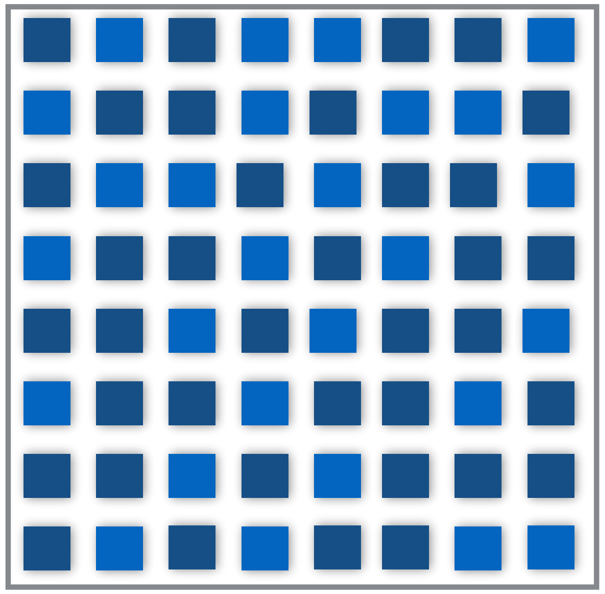

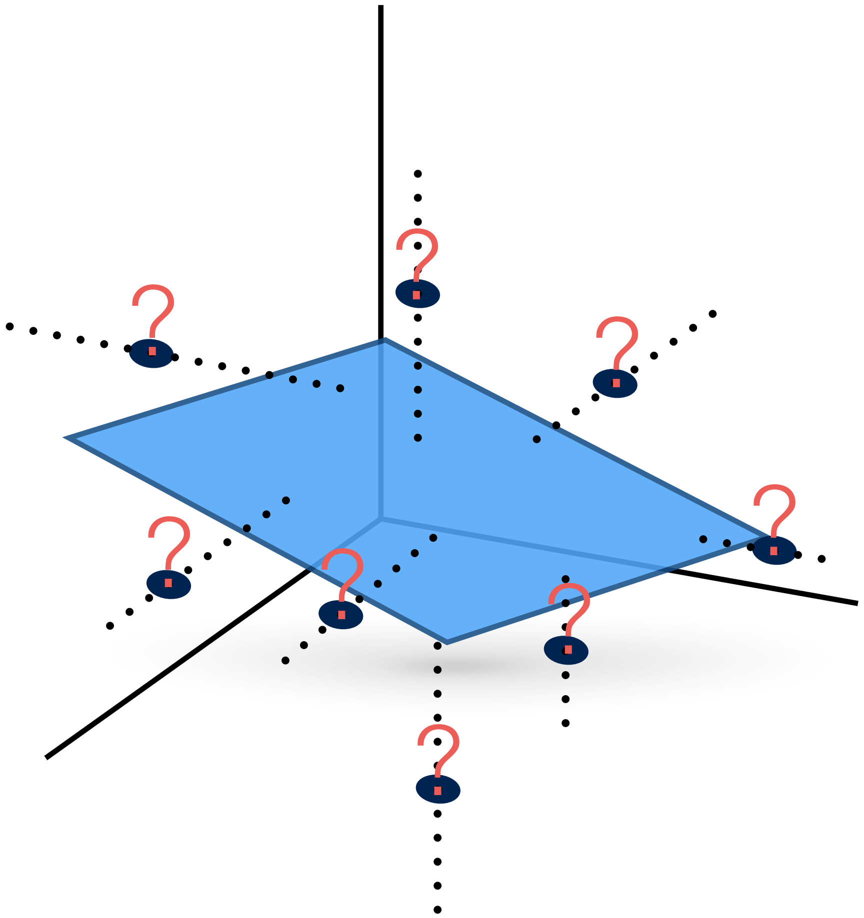

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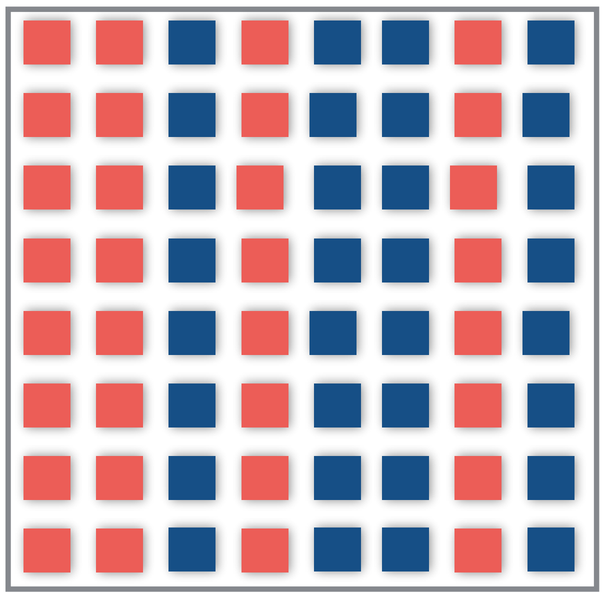

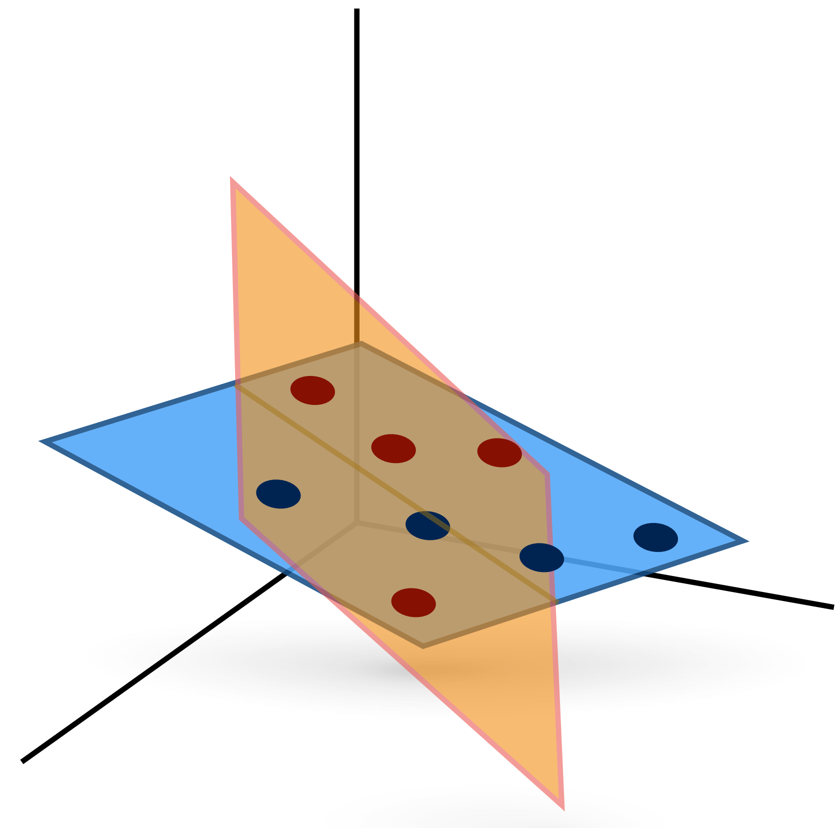

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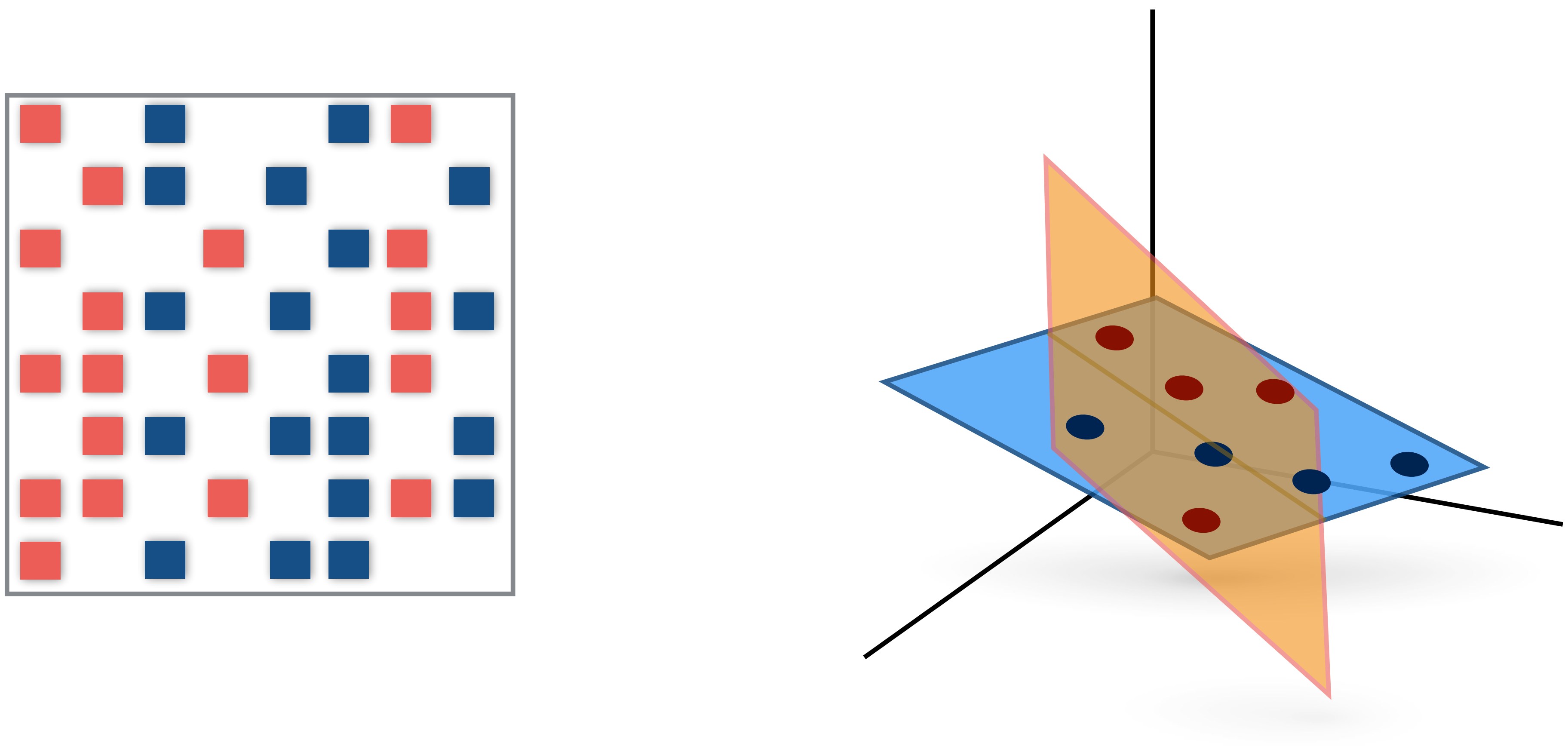

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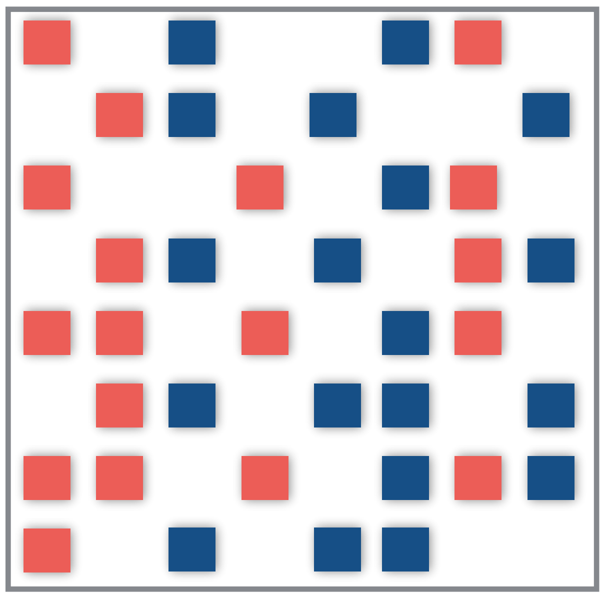 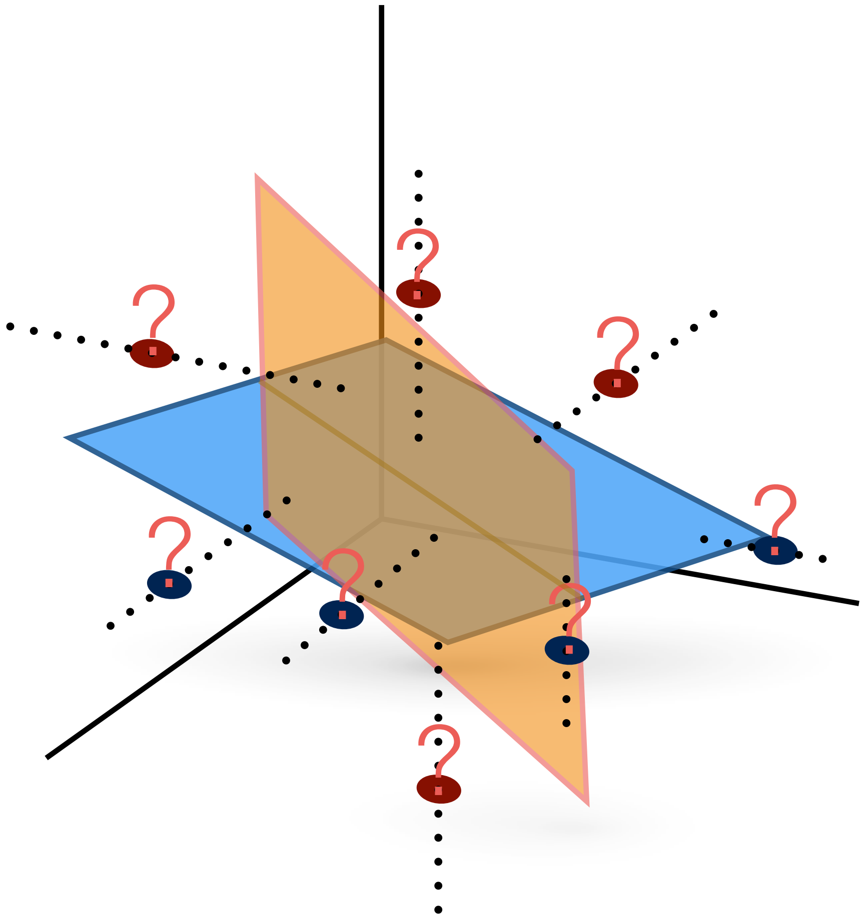

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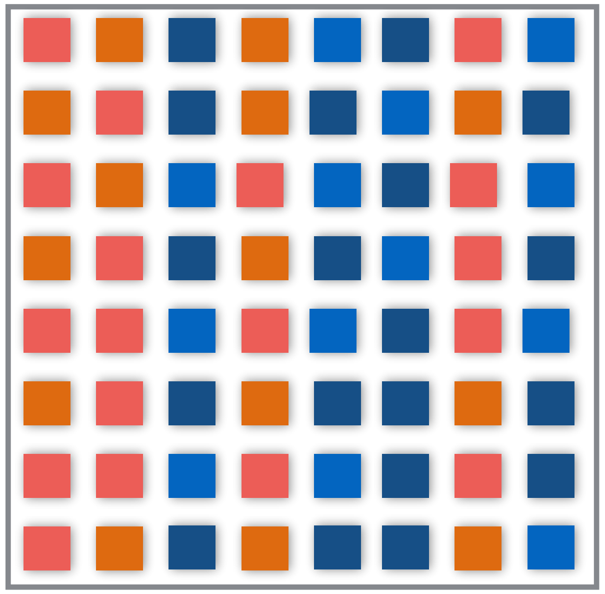

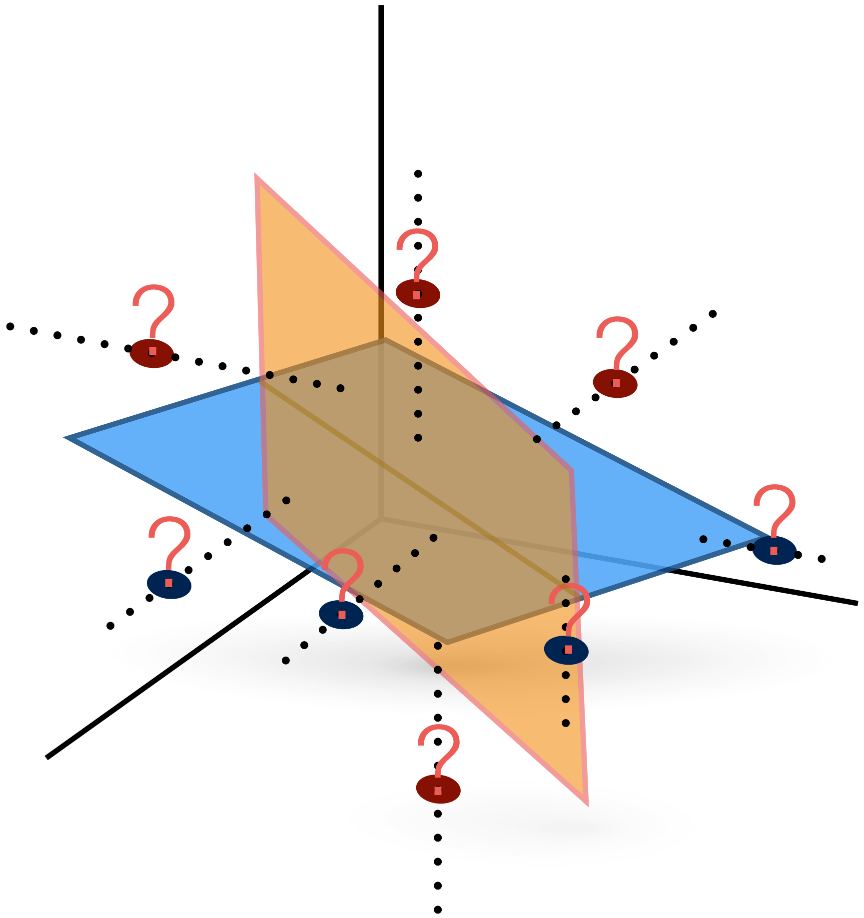

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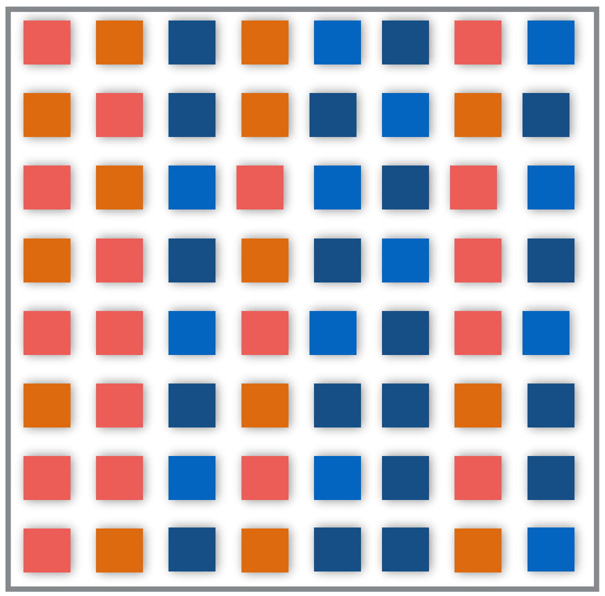
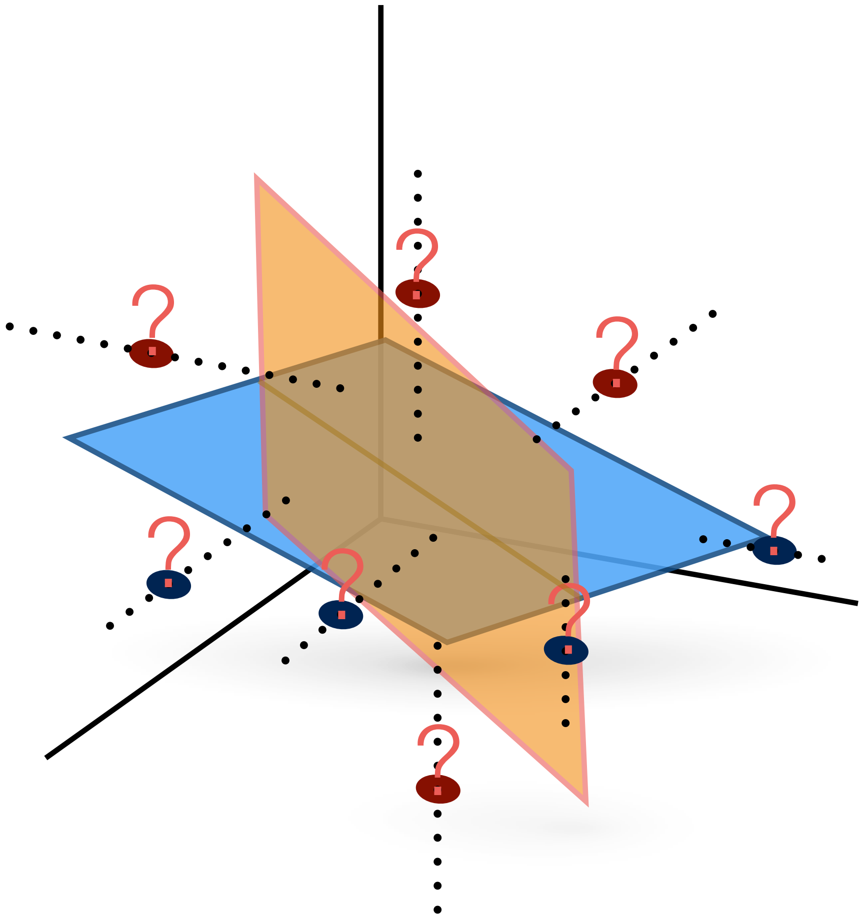
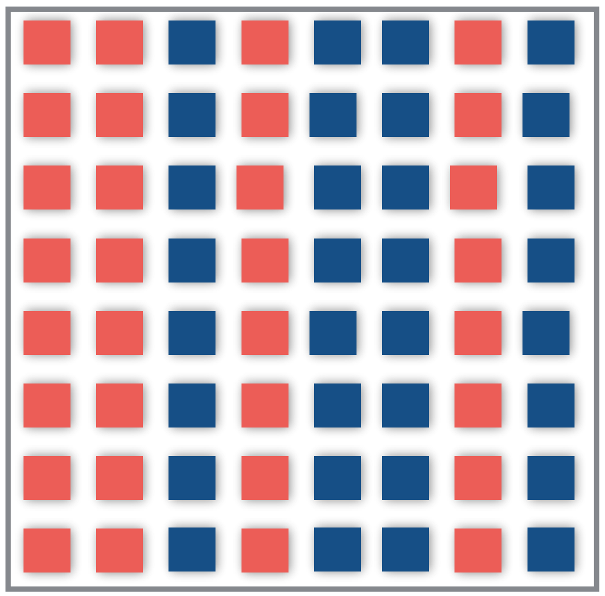
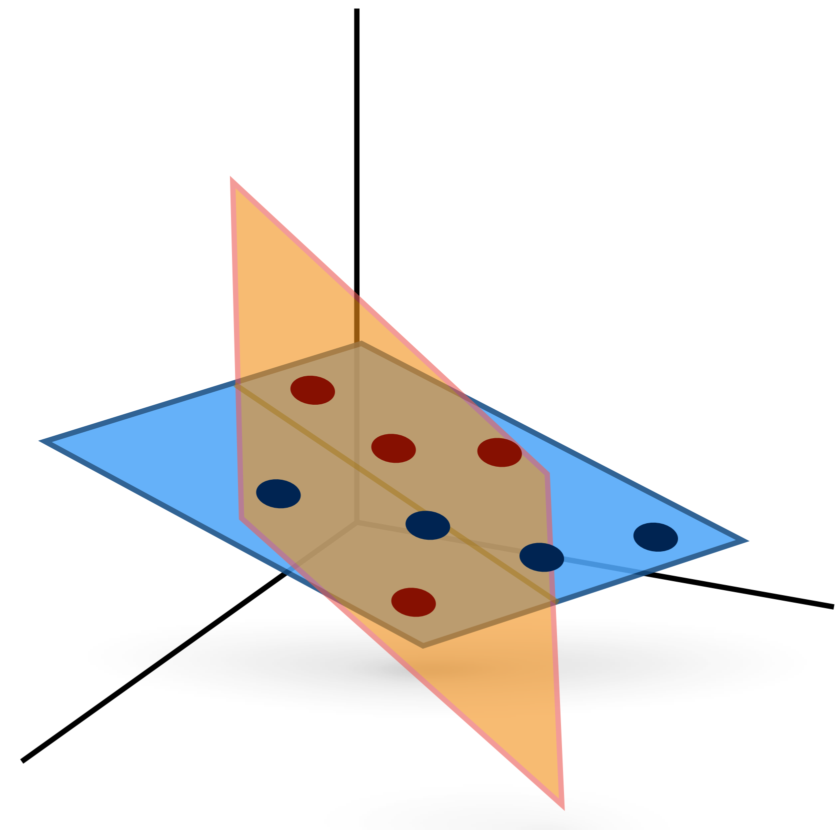




Mixture Matrix Completion

a.k.a. High-Rank Matrix Completion or
Subspace Clustering with Missing Data.





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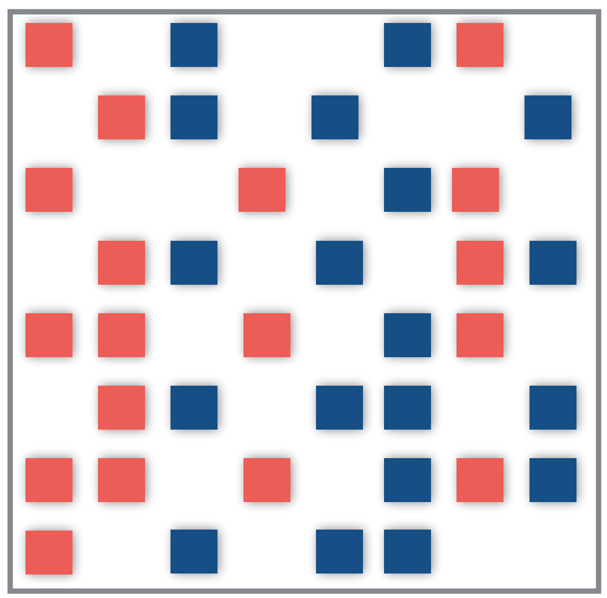

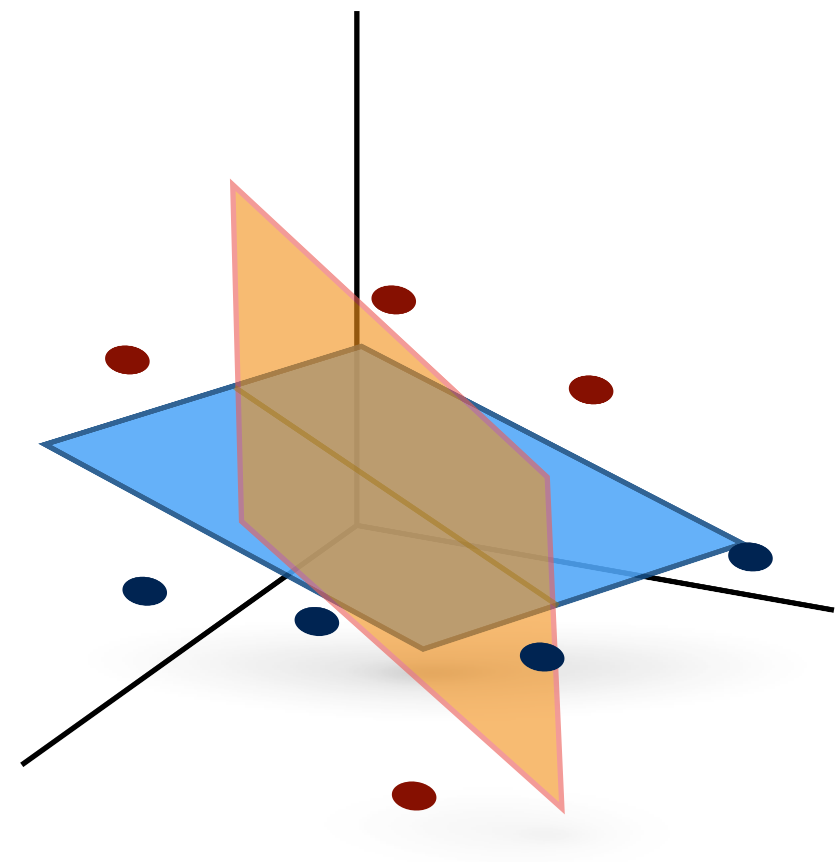

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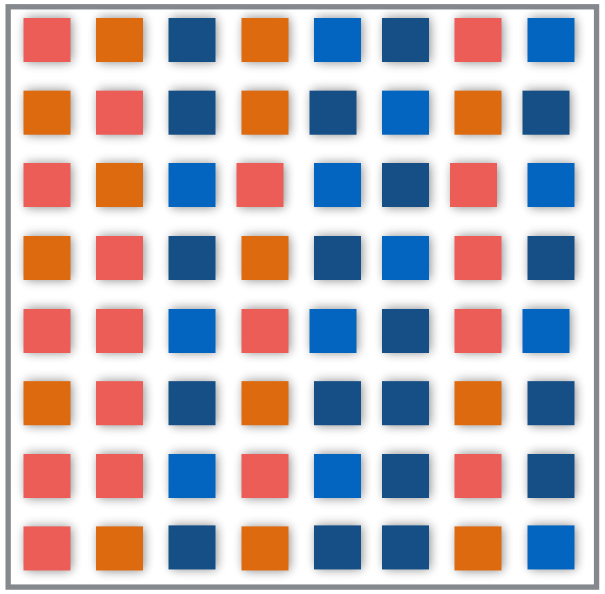 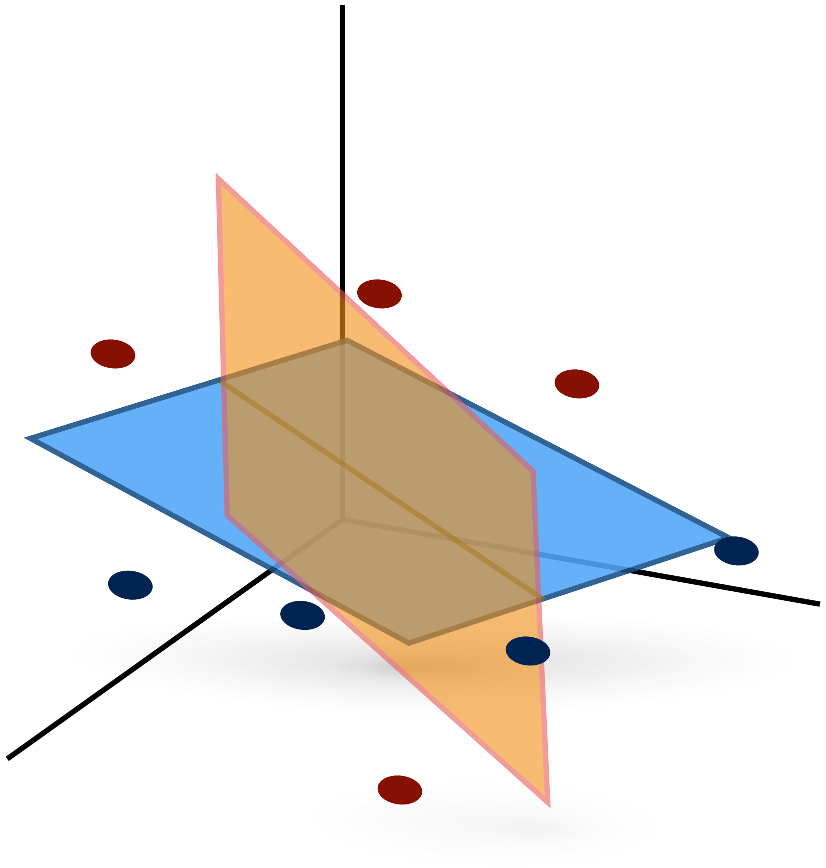

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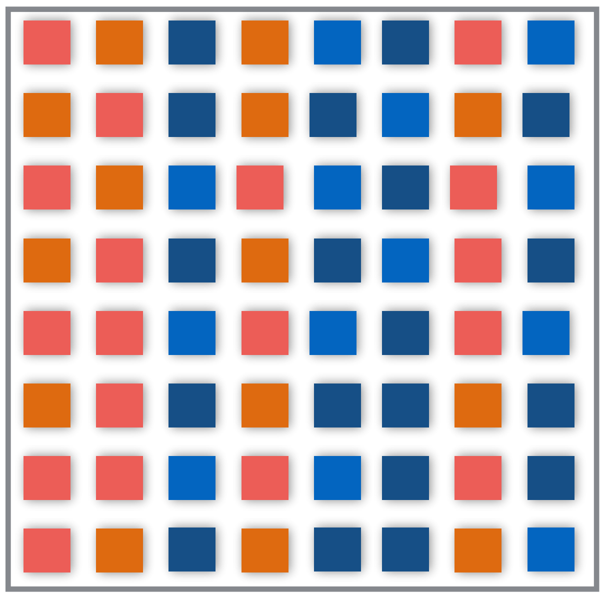

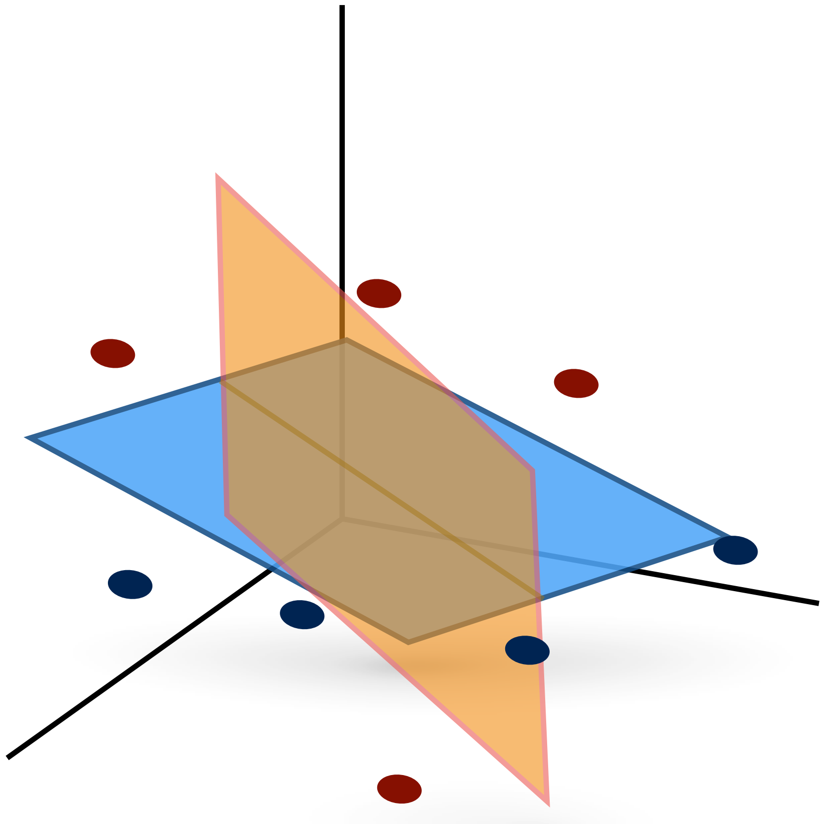

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a.k.a. High-Rank Matrix Completion or
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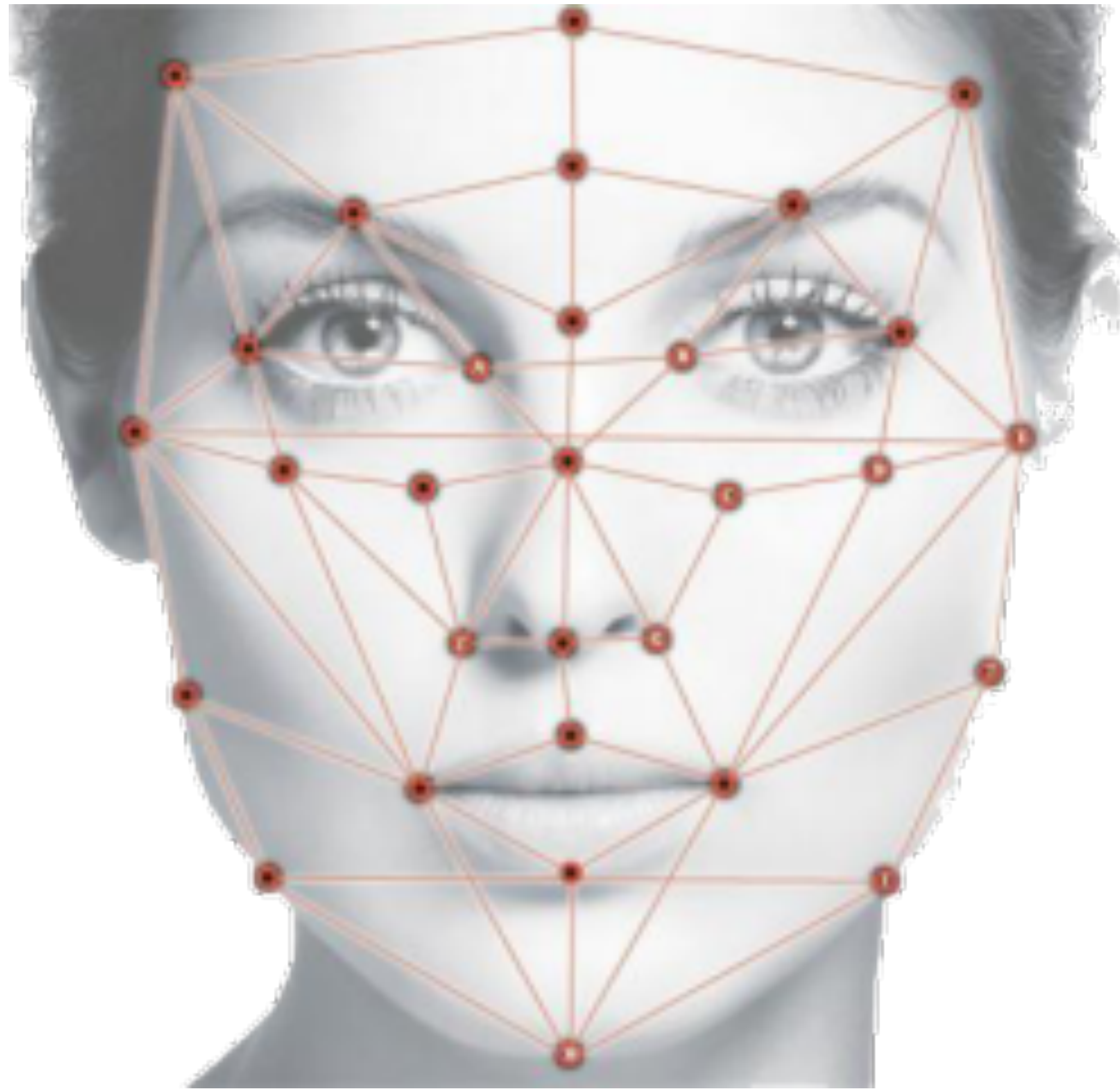
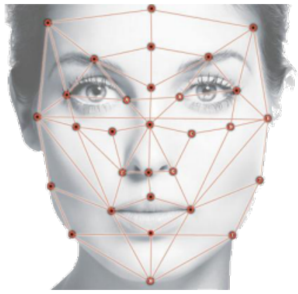
What is this good for?



Lots of Applications



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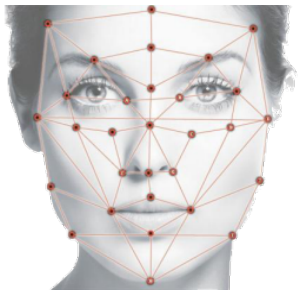

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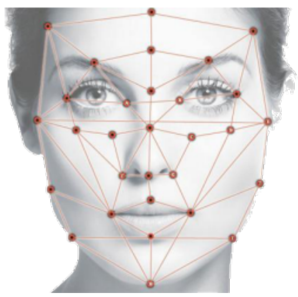

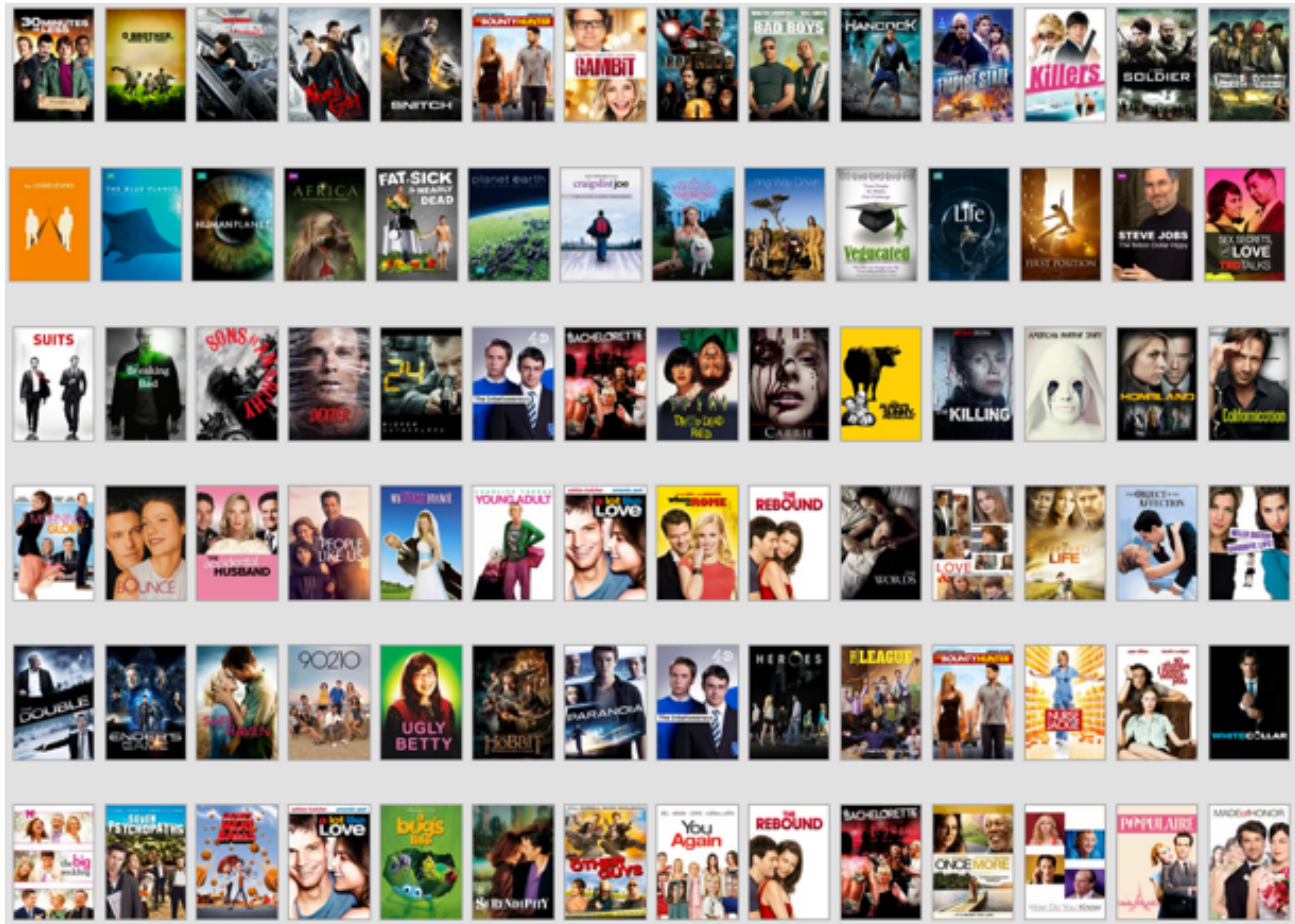

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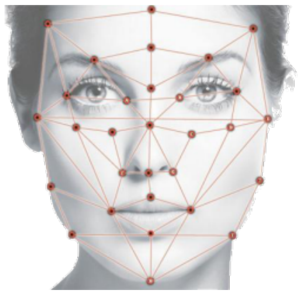

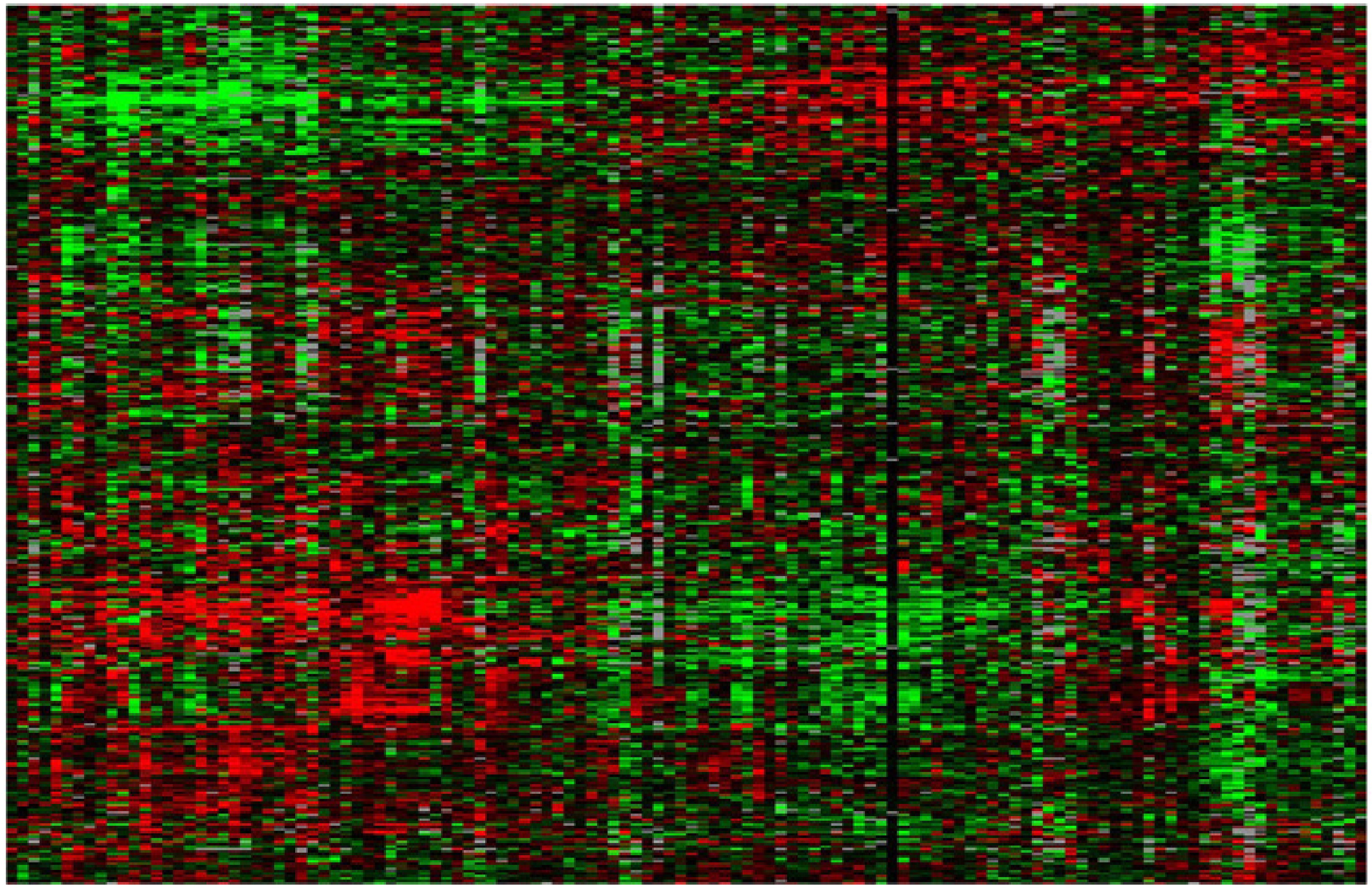

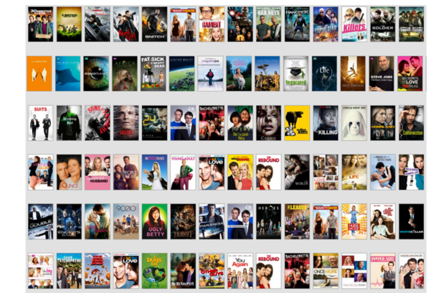

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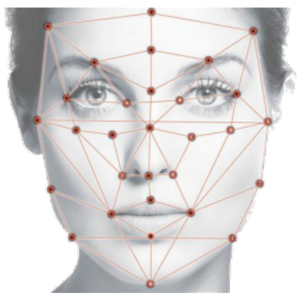



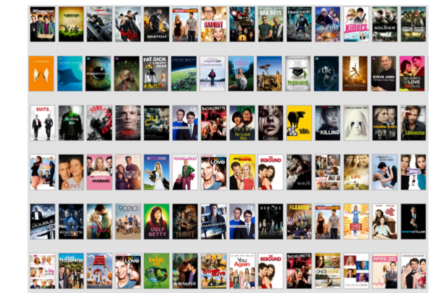
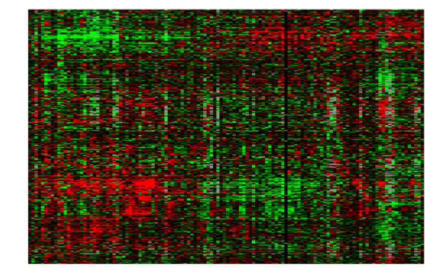

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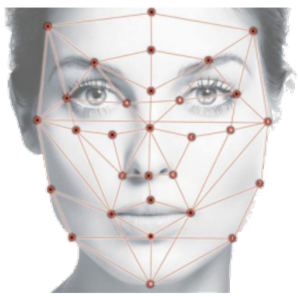

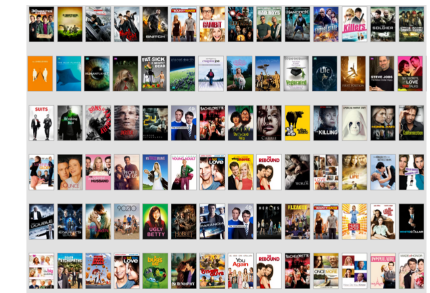

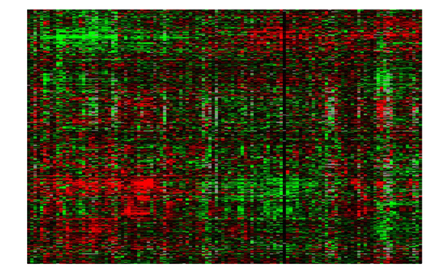

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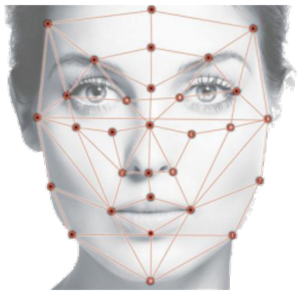
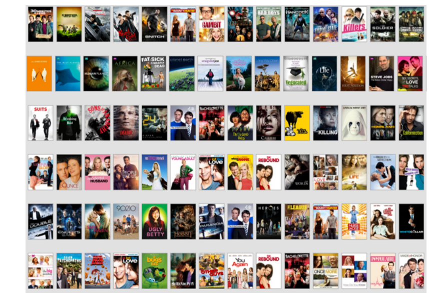
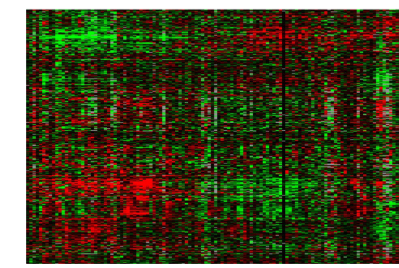
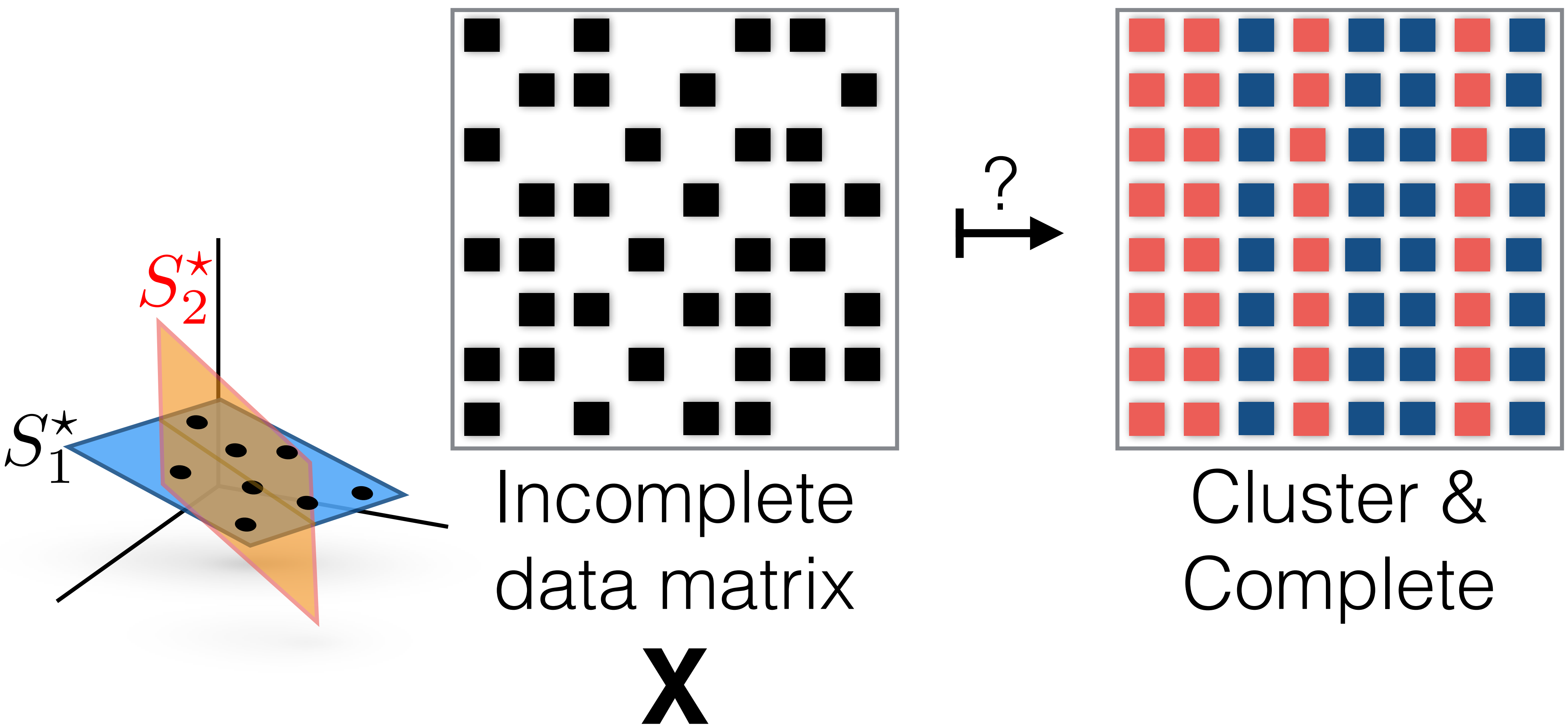






Lots of Applications

In general



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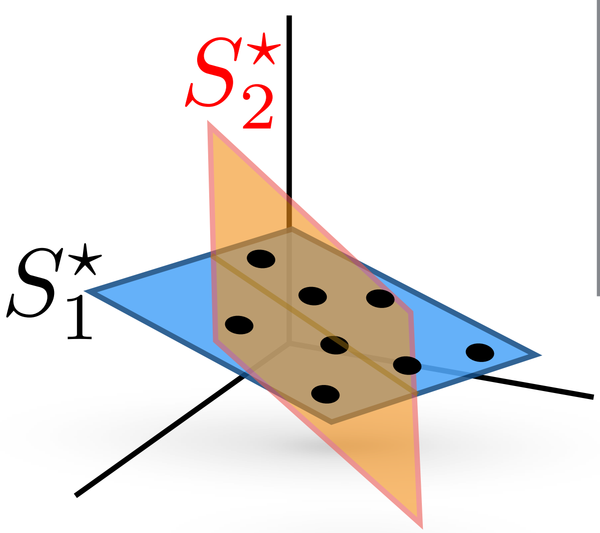
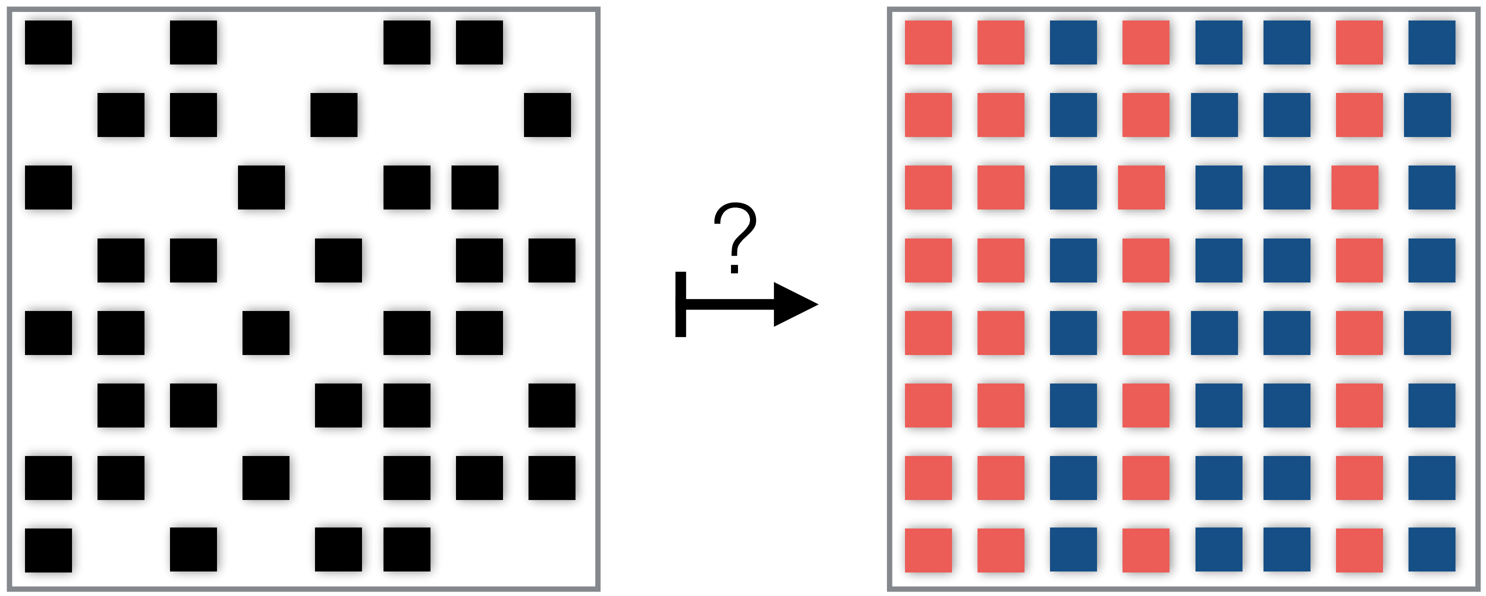




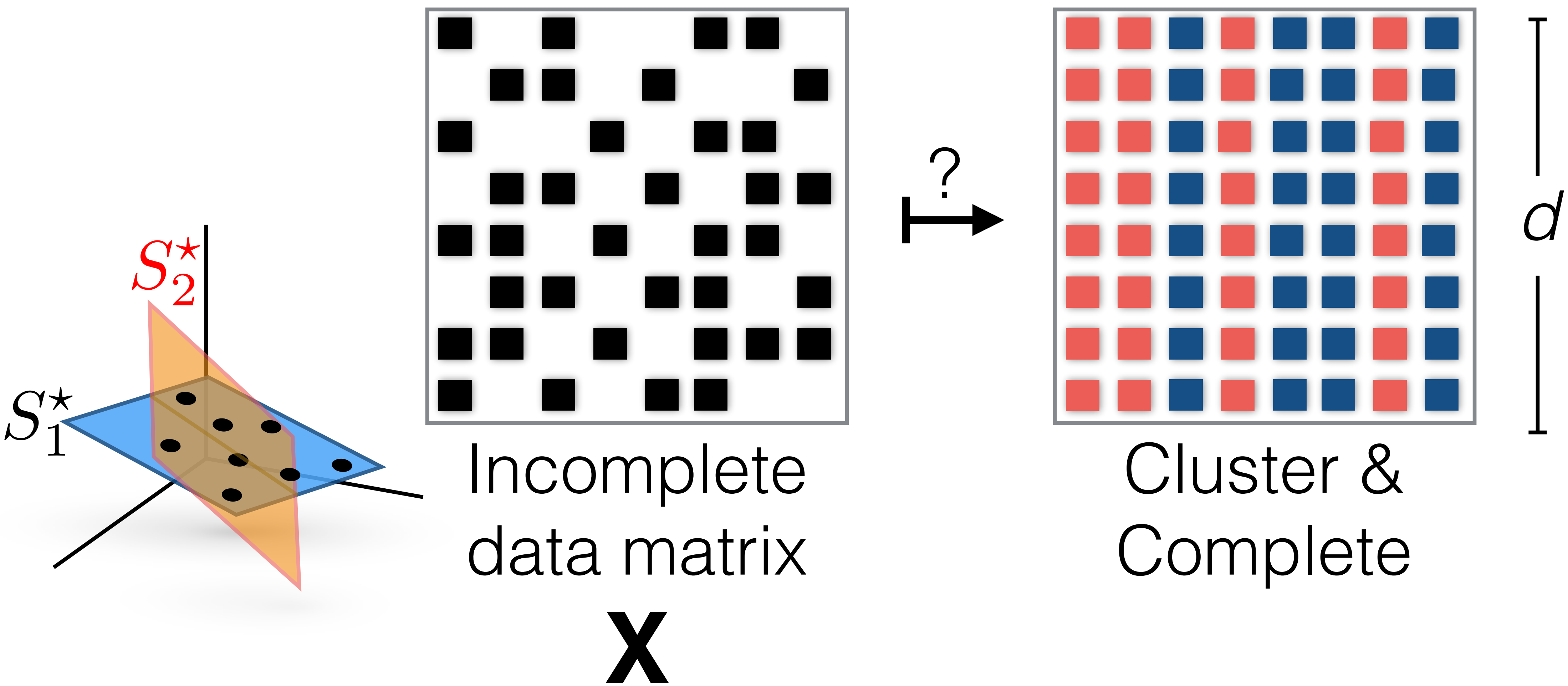
Incomplete data matrix

X

Cluster & Complete

Can this be done? When? How?

In general



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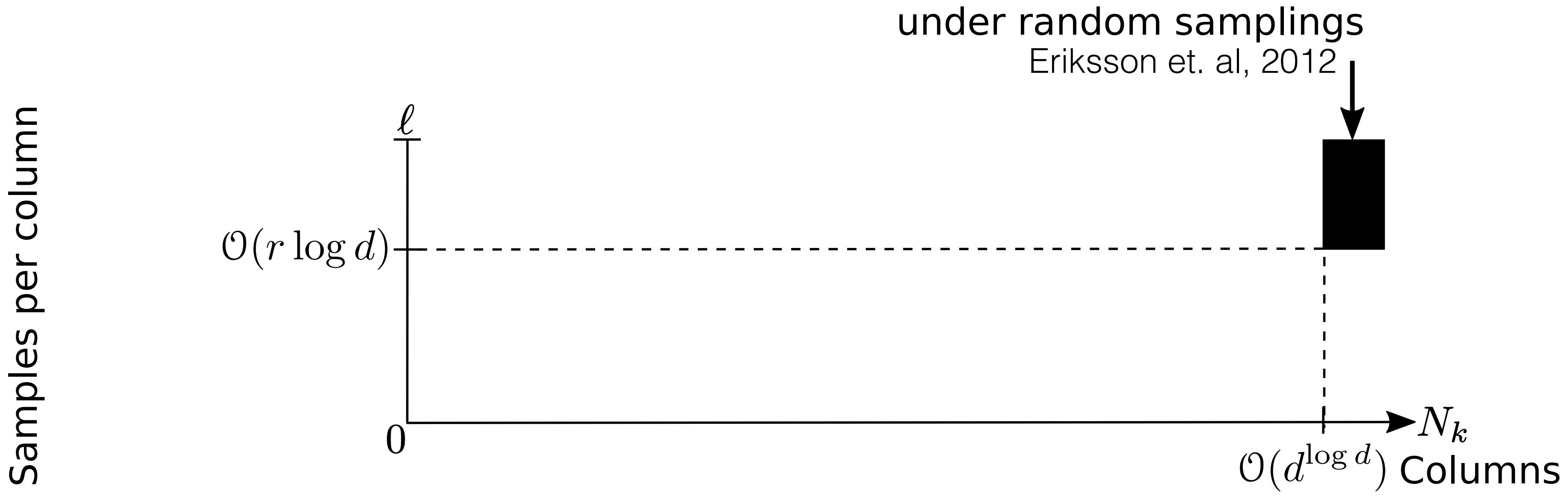

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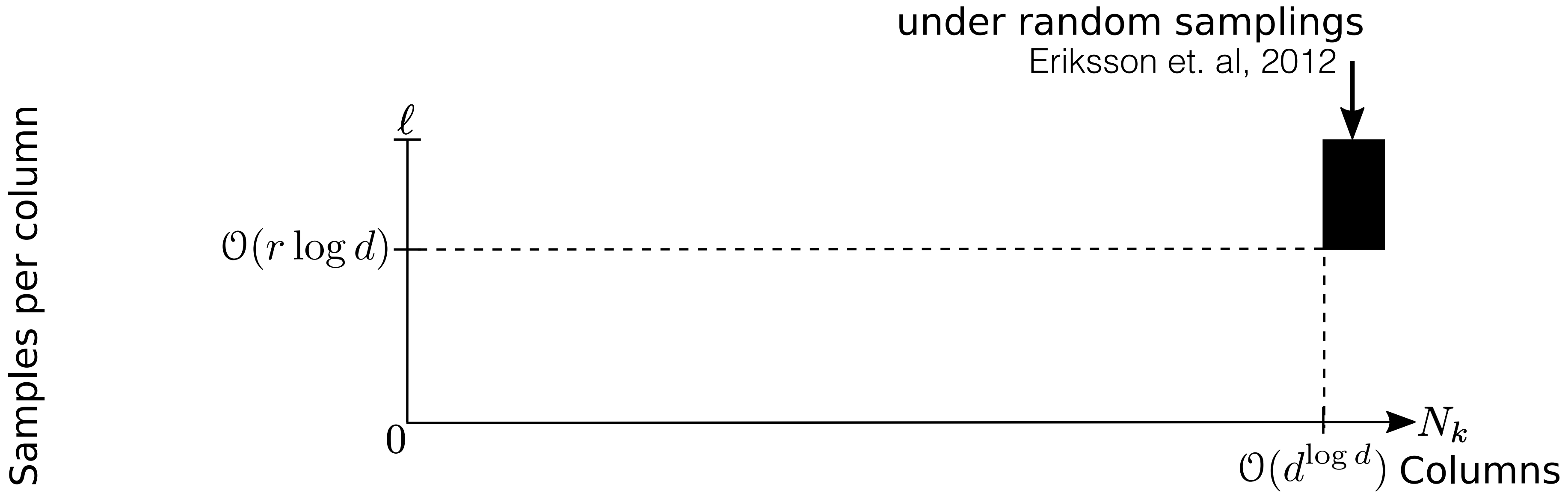

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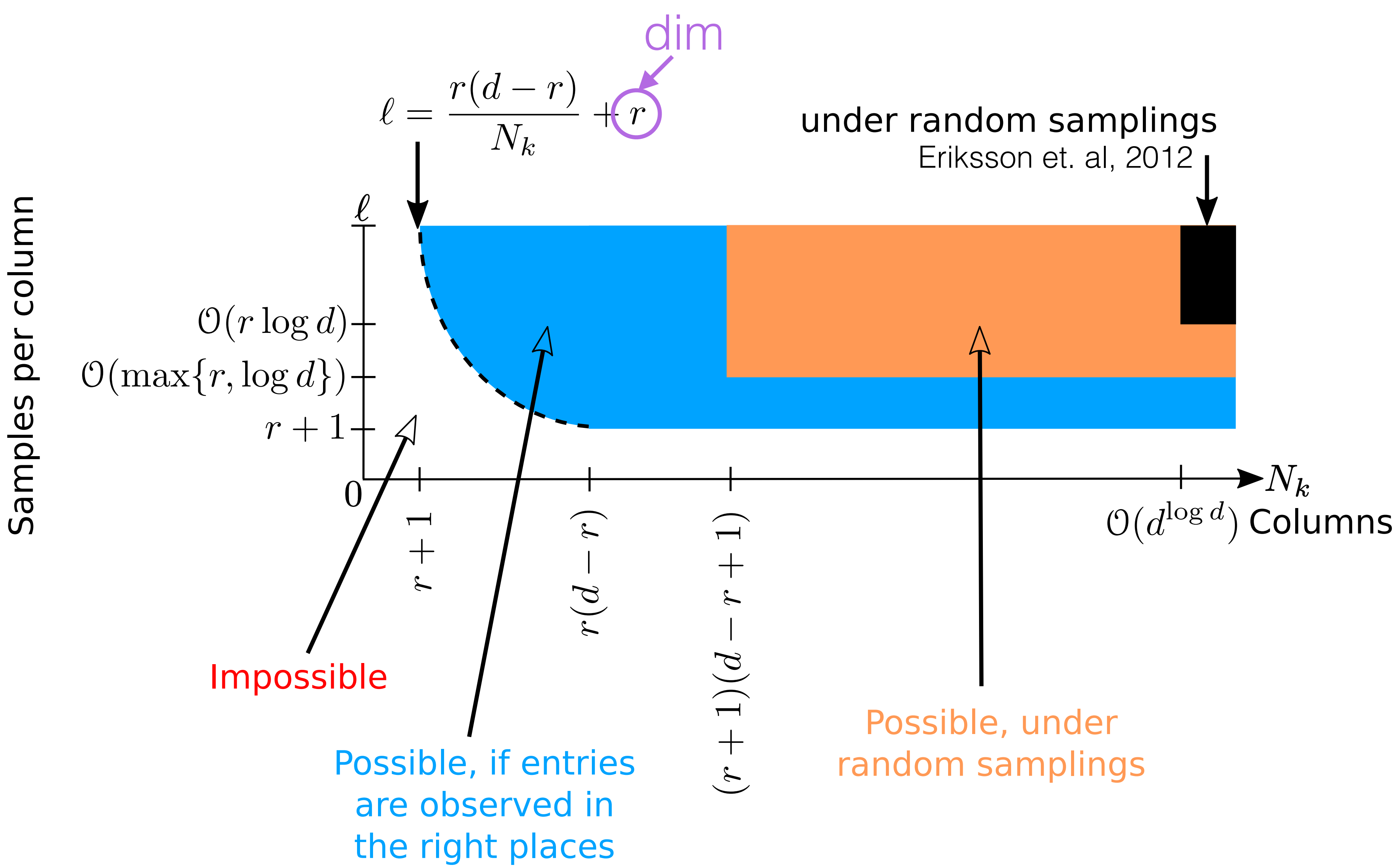
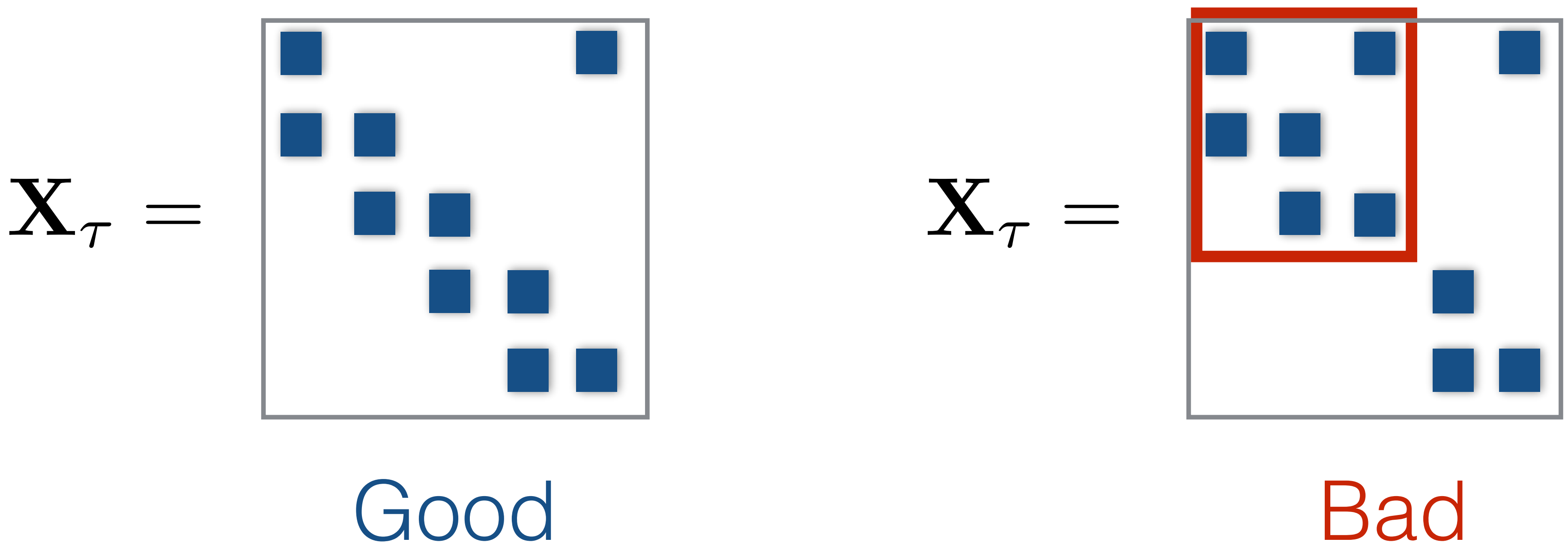


Information-theoretic requirements

P.-A., Nowak, 2016

Let $\mathbf{X}_\tau$ be a matrix formed with $d - r + 1$ columns of $\mathbf{X}_k$. We say $\mathbf{X}_\tau$ is observed in the right places if every proper subset of n columns of $\mathbf{X}_\tau$ has observations on at least $n + r$ rows.



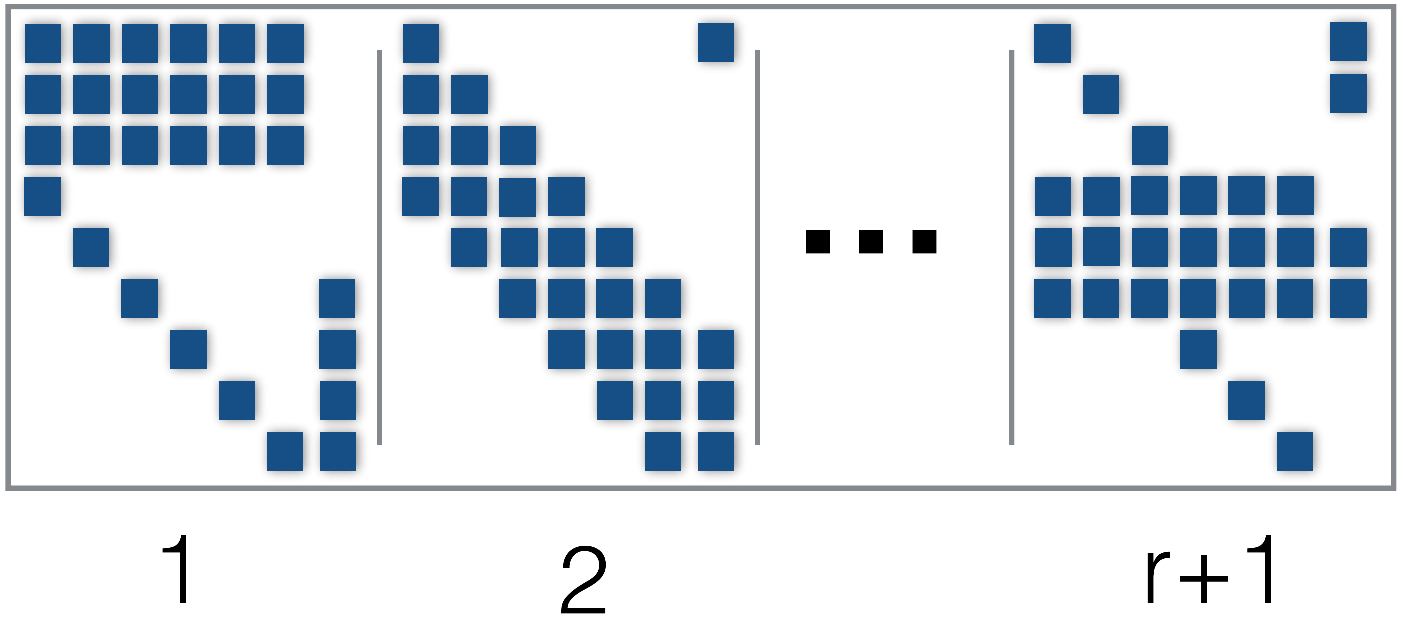
Observed in the right places

P.-A., Nowak, 2016

What do I mean?

For every k, suppose $\mathbf{X}_k$ contains $r+1$ disjoint matrices $\{\mathbf{X}_\tau\}_{\tau=1}^{r+1}$ observed in the right entries. Then $\{S_k^\star\}_{k=1}^{K}$ is the only union of subspaces that agrees with $\mathbf{X}$.

$\mathbf{X}_k =$



Information-theoretic requirements

P.-A., Nowak, 2016

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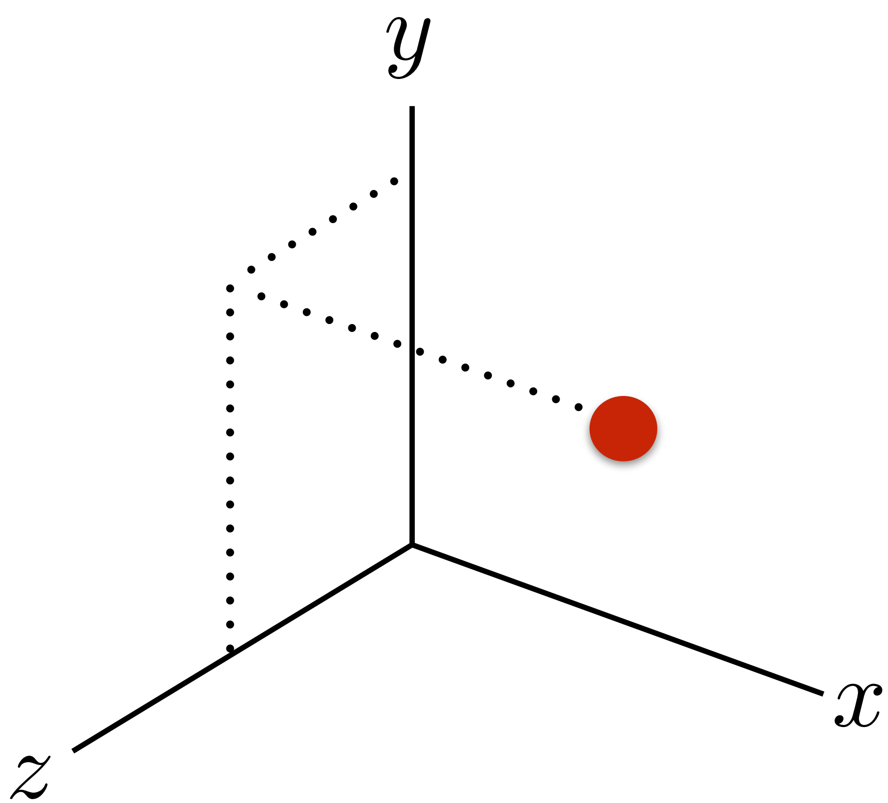
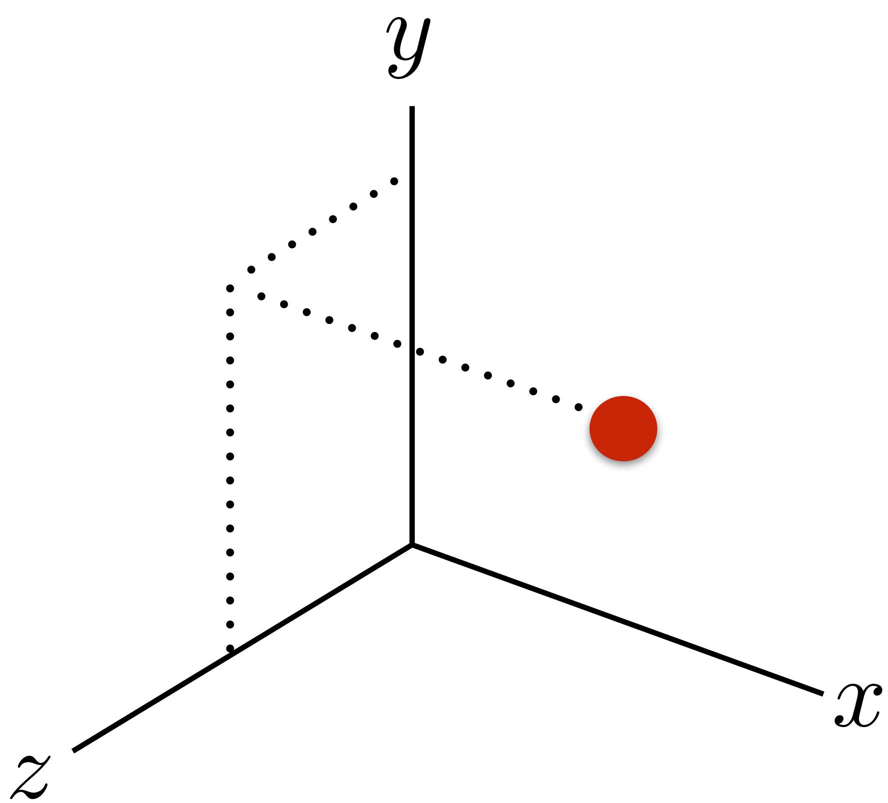


$$\begin{bmatrix} \cdot \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas



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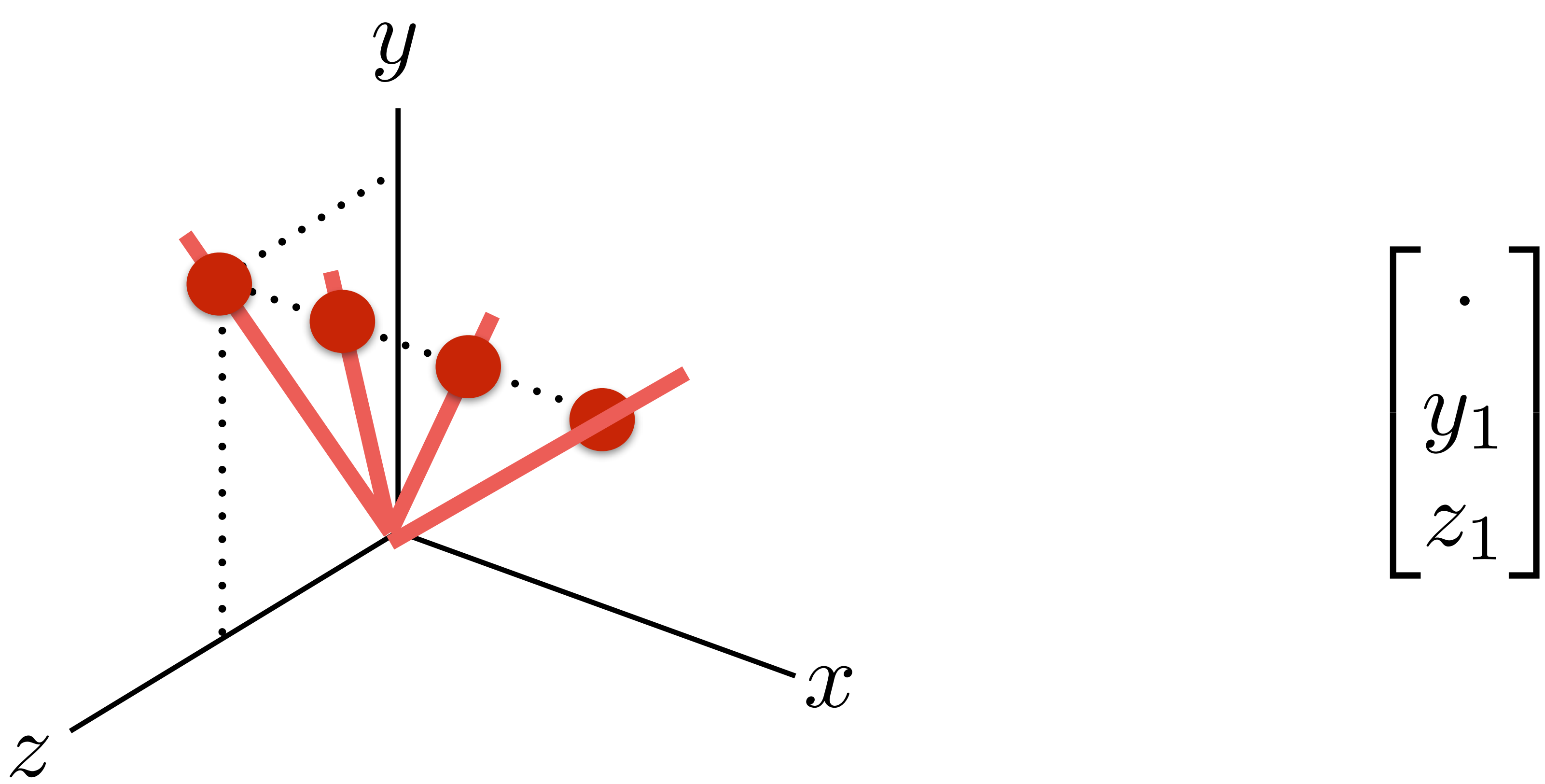

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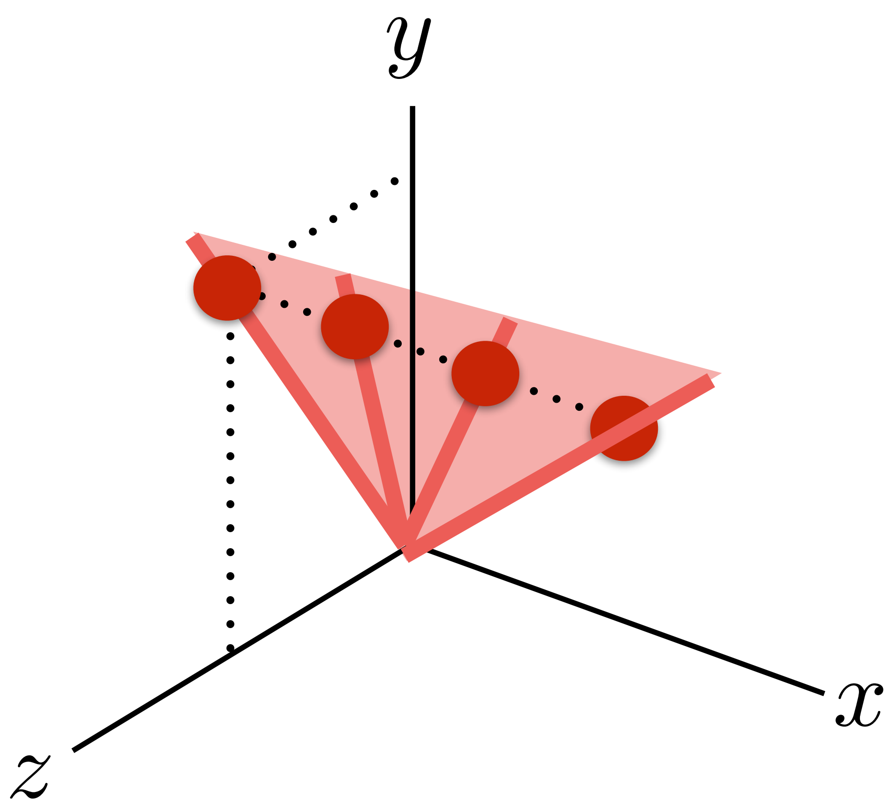
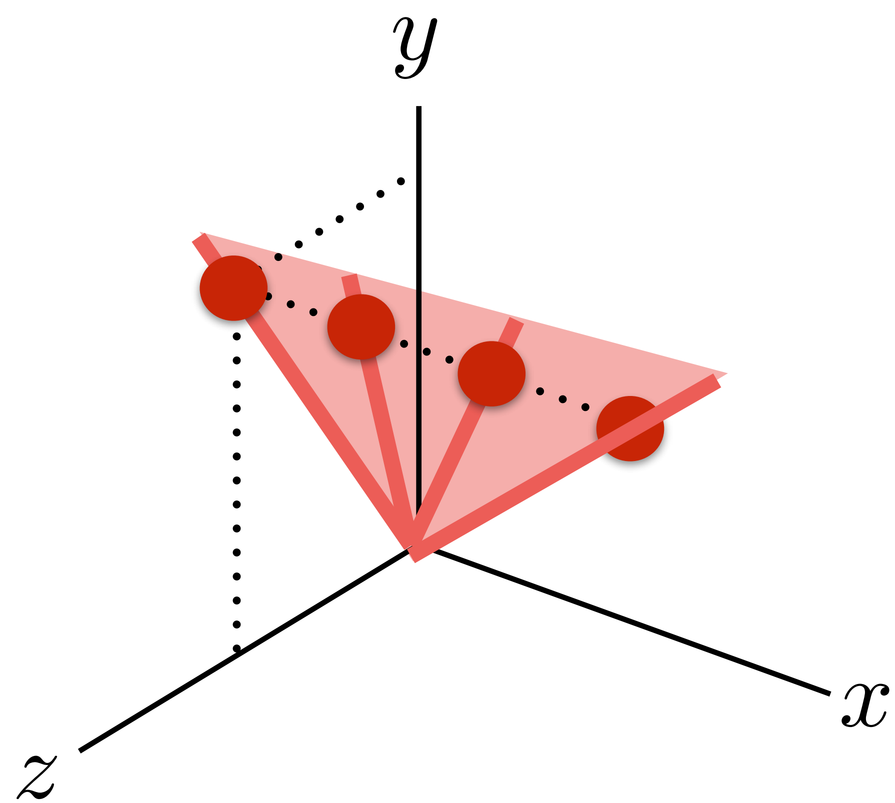


$$\begin{bmatrix} \cdot \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas

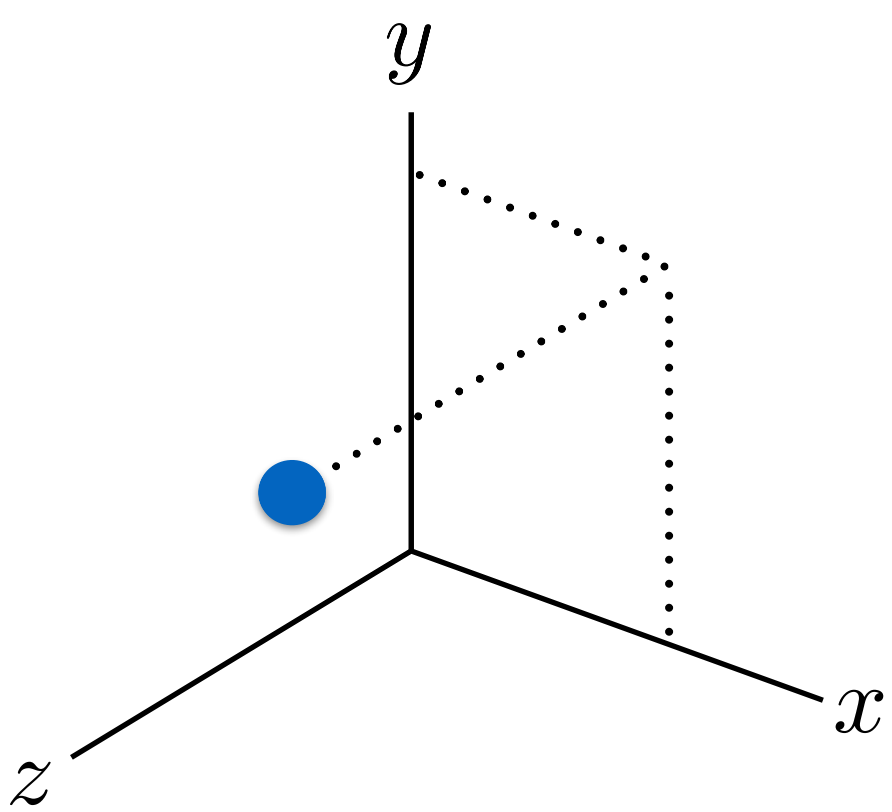
This column imposes *1* restriction on what the subspace may be



$$\begin{bmatrix} \cdot \\ y_1 \\ z_1 \end{bmatrix}$$

A flavor of our ideas

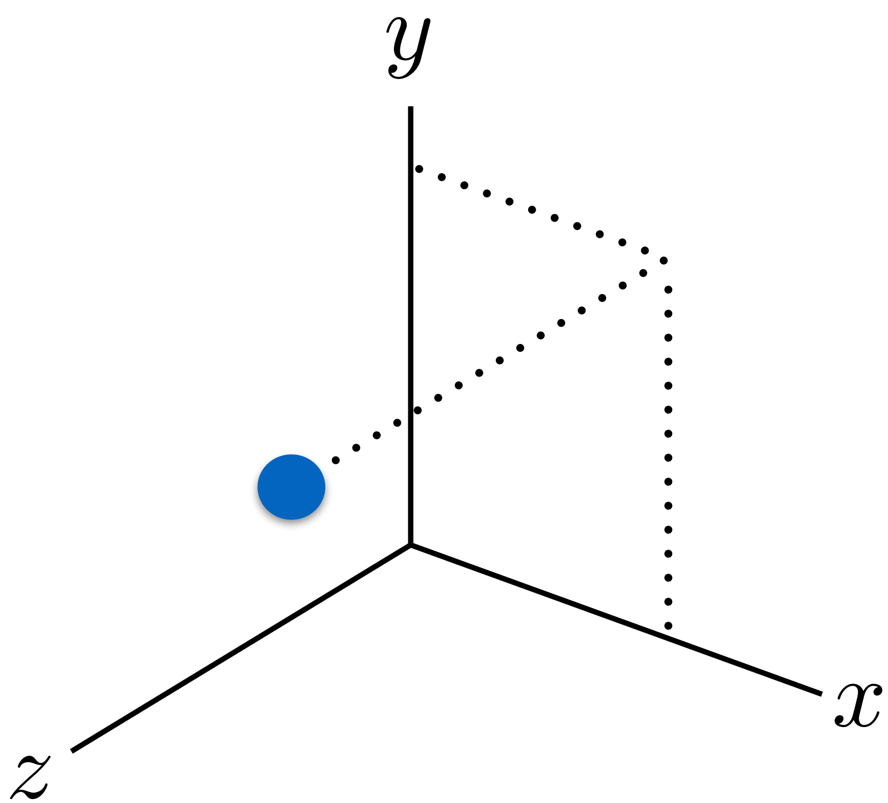
New column imposes *1* more restriction on what the subspace may be



$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

A flavor of our ideas

New column imposes 1 more restriction on what the subspace may be



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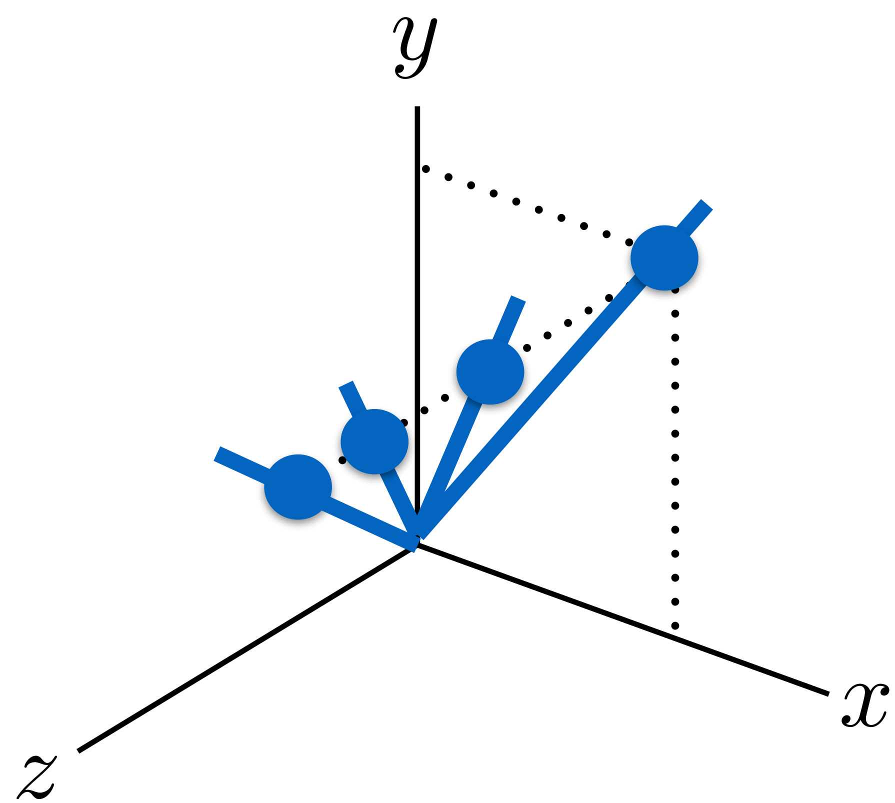

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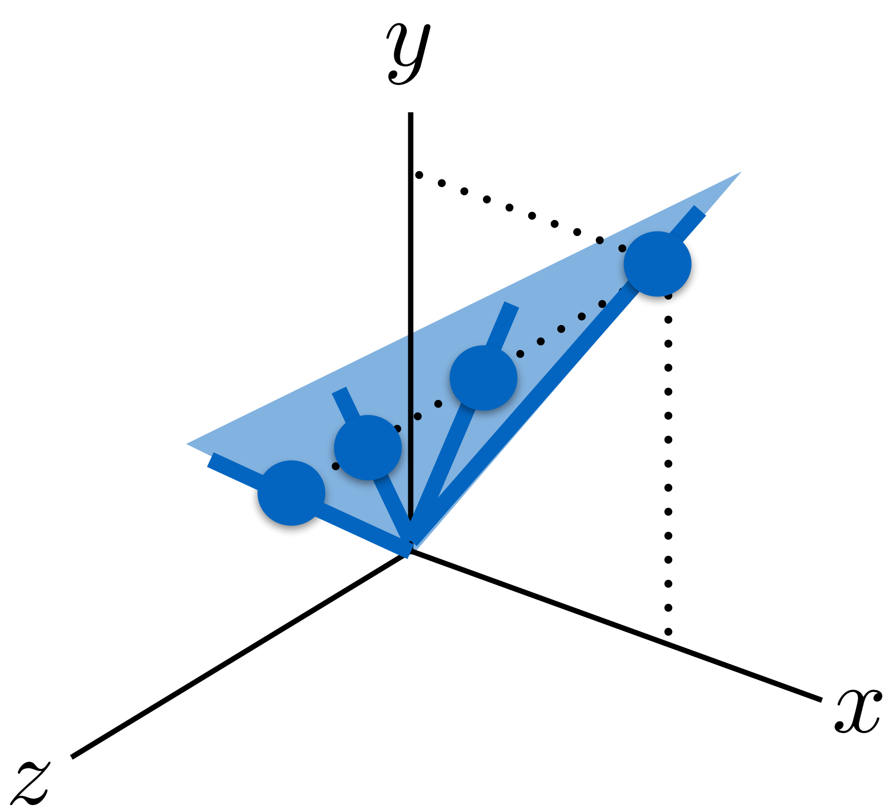

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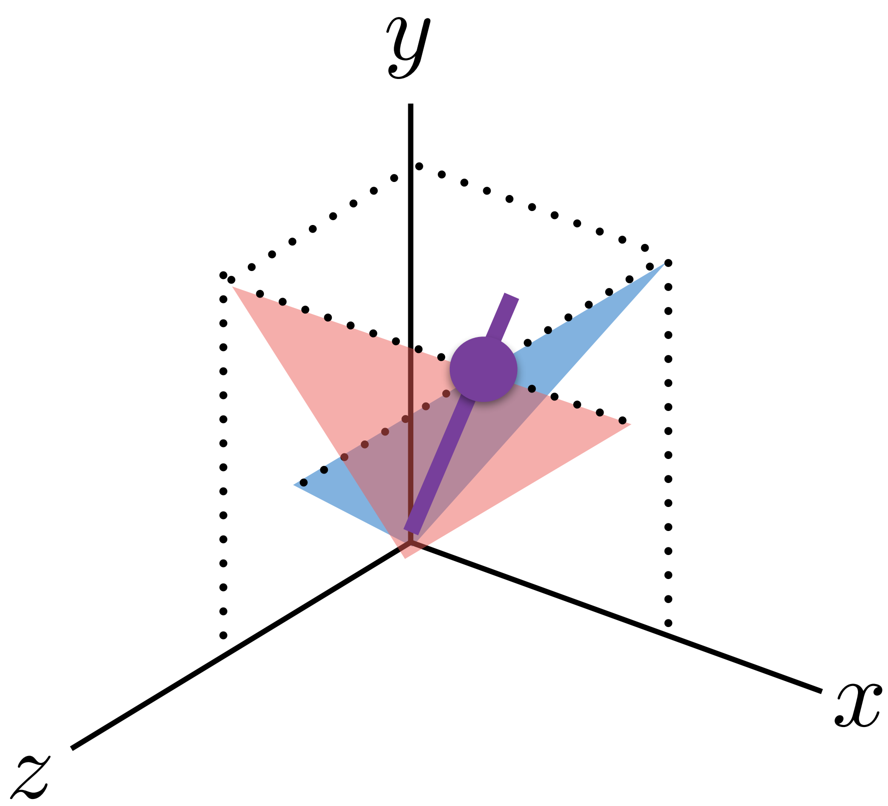
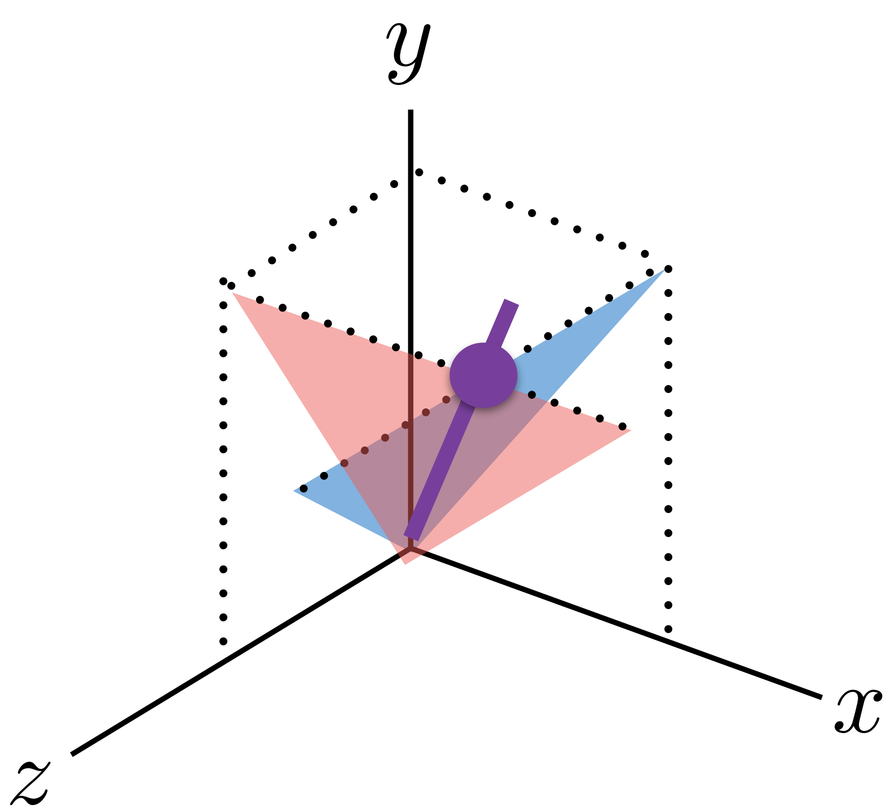


Each column with *r+1* entries imposes *1* restriction on what the subspace may be

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

A flavor of our ideas



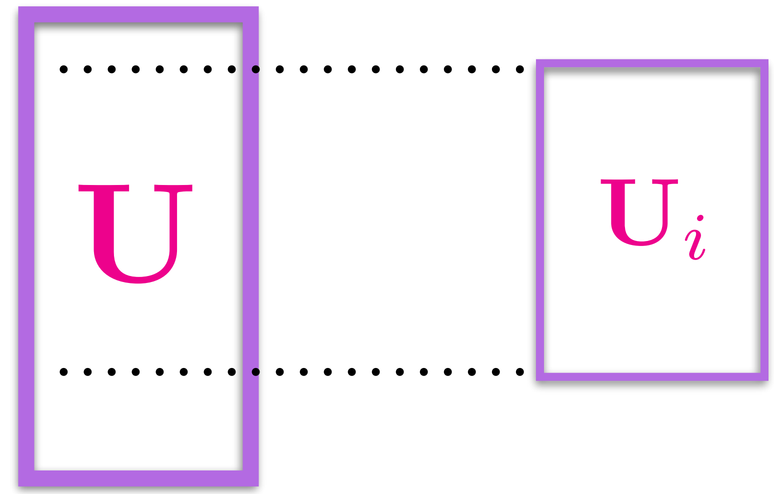
Each column with *r+1* entries imposes *1* restriction on what the subspace may be

Degree-*r* polynomial

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

A flavor of our ideas

Take a basis of an arbitrary subspace



This subspace agrees with $\mathbf{x}_i$ if and only if

$$\mathbf{x}_i = \mathbf{U}_i \boldsymbol{\theta}_i$$

A flavor of our ideas

- We can split this as:

$$\begin{matrix} r\left\{ \right. \\ 1\left\{ \right. \end{matrix} \left[\begin{array}{c} \boldsymbol{x}_{\boldsymbol{\Delta}_i} \\ \hline \boldsymbol{x}_{\boldsymbol{\nabla}_i} \end{array} \right] = \left[\begin{array}{c} \mathbf{U}_{\boldsymbol{\Delta}_i} \\ \hline \mathbf{U}_{\boldsymbol{\nabla}_i} \end{array} \right] \boldsymbol{\theta}_i.$$

- We can use the top block to solve for $\boldsymbol{\theta}_i$:

$$\boldsymbol{\theta}_i = \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}.$$

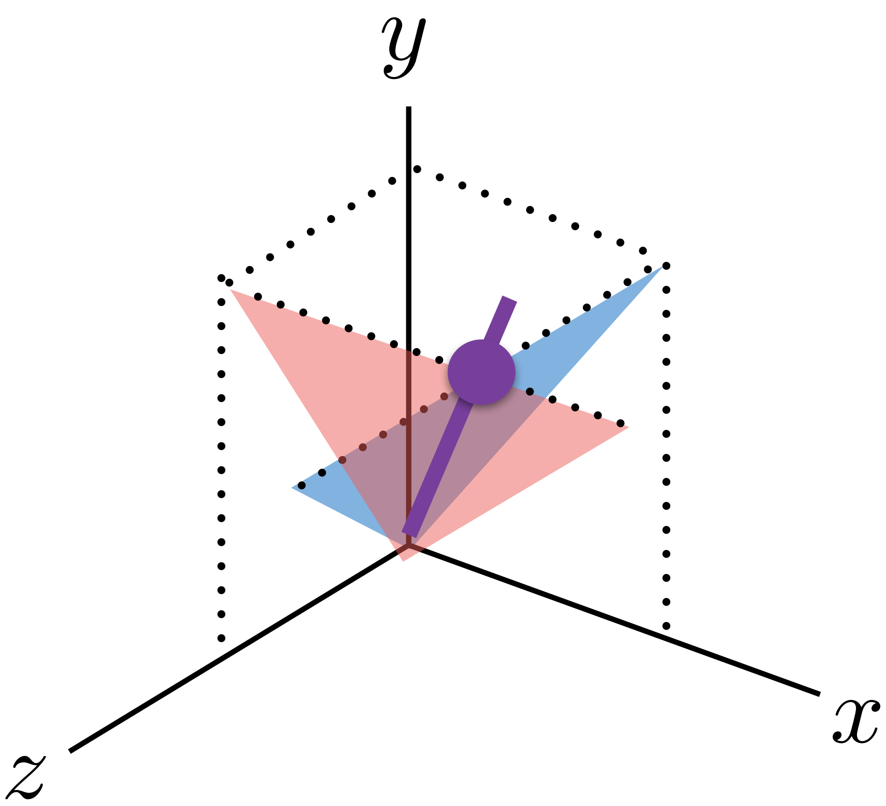
- Plug this in the last row:

$$\boldsymbol{x}_{\boldsymbol{\nabla}_i} = \mathbf{U}_{\boldsymbol{\nabla}_i} \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}.$$

- Or equivalently

$$\underbrace{\boldsymbol{x}_{\boldsymbol{\nabla}_i} - \mathbf{U}_{\boldsymbol{\nabla}_i} \mathbf{U}_{\boldsymbol{\Delta}_i}^{-1} \boldsymbol{x}_{\boldsymbol{\Delta}_i}}_{f_i(\mathbf{U}_i | x_i)} = 0.$$

A flavor of our ideas



$$\mathbf{X} = \begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

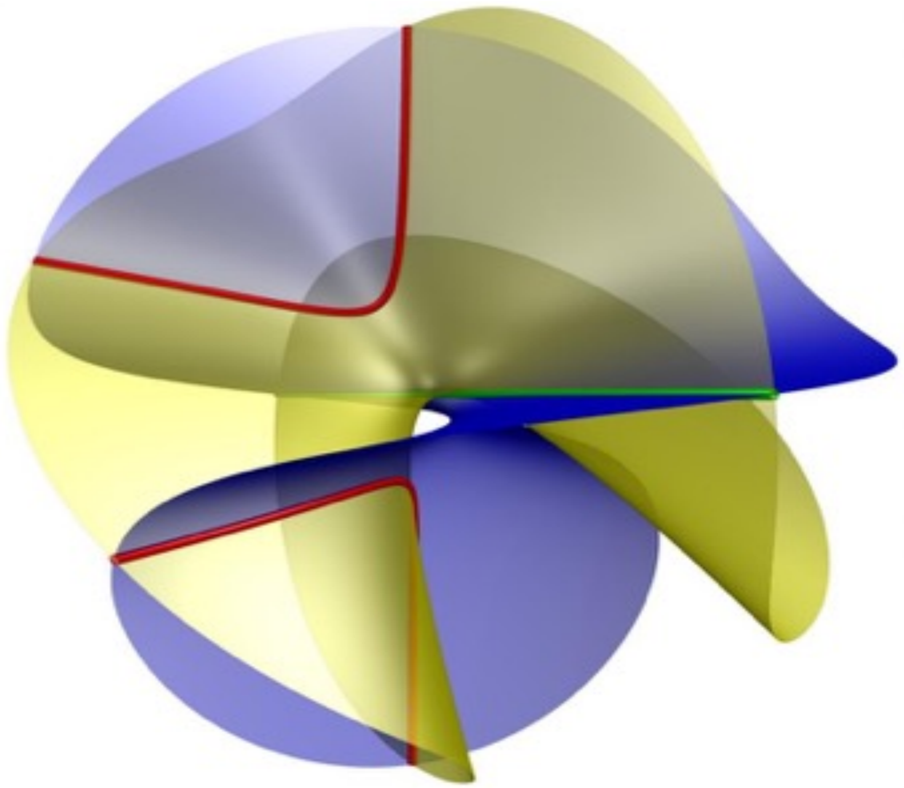
A subspace S agrees with $\mathbf{X}$

$$\Updownarrow$$

$$f_1(\mathbf{U}_1|x_1) = 0$$

$$f_2(\mathbf{U}_2|x_2) = 0$$

A flavor of our ideas



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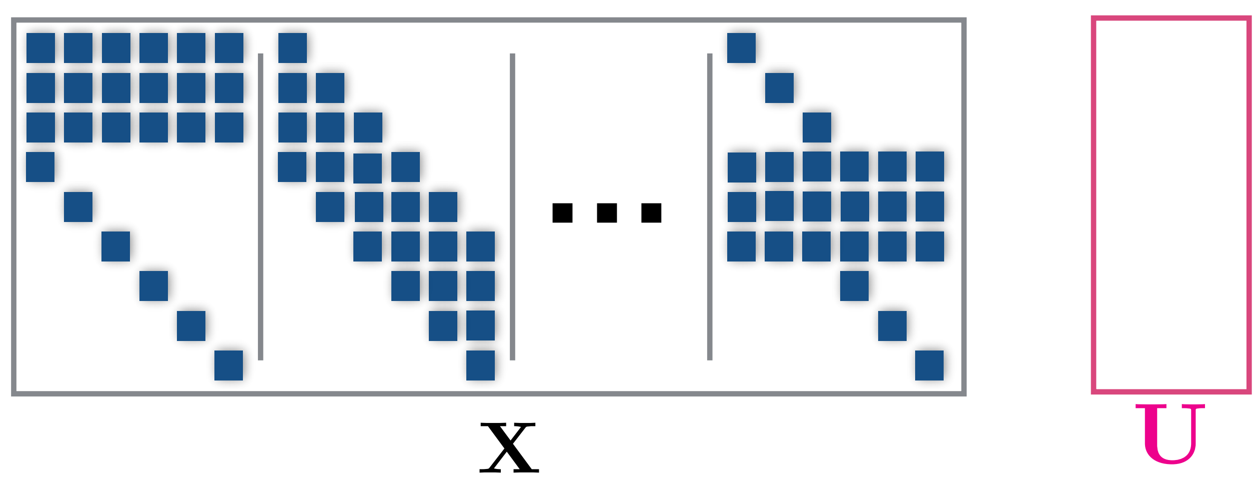
A flavor of our ideas

- Each of column produces one polynomial

$$f_1(\mathbf{U}_1|x_1),\ f_2(\mathbf{U}_2|x_2),\ \ldots,\ f_N(\mathbf{U}_N|x_N)$$

- The observed rows indicate the variables involved
- If data is observed in *the right entries*, all variables will be pined down



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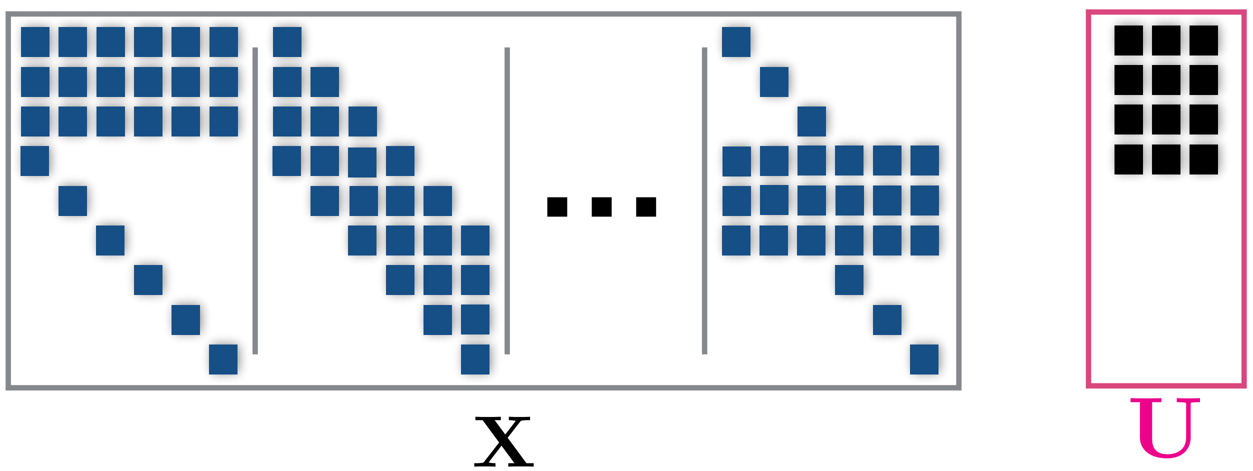

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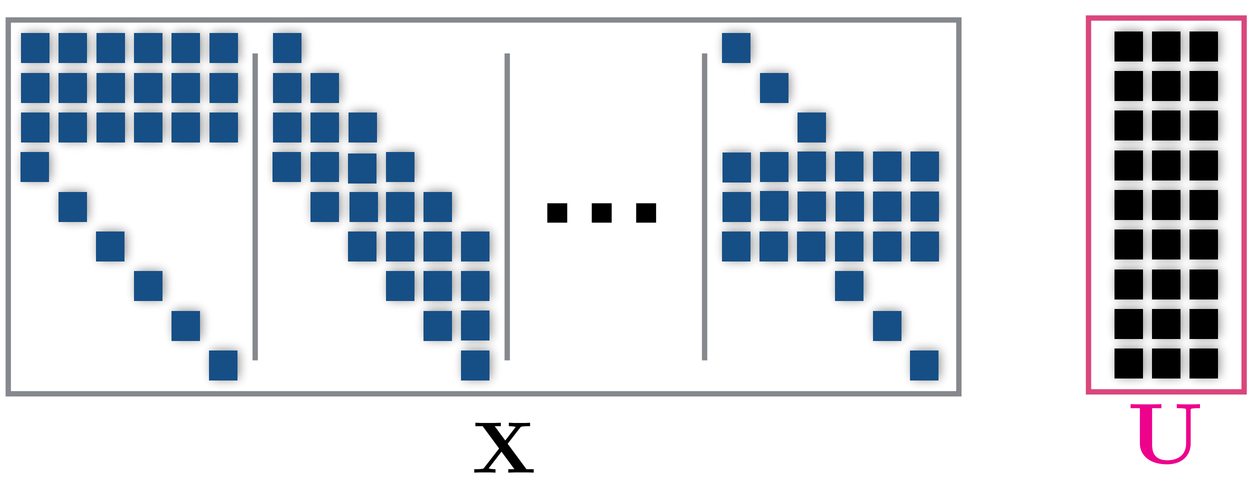

A flavor of our ideas

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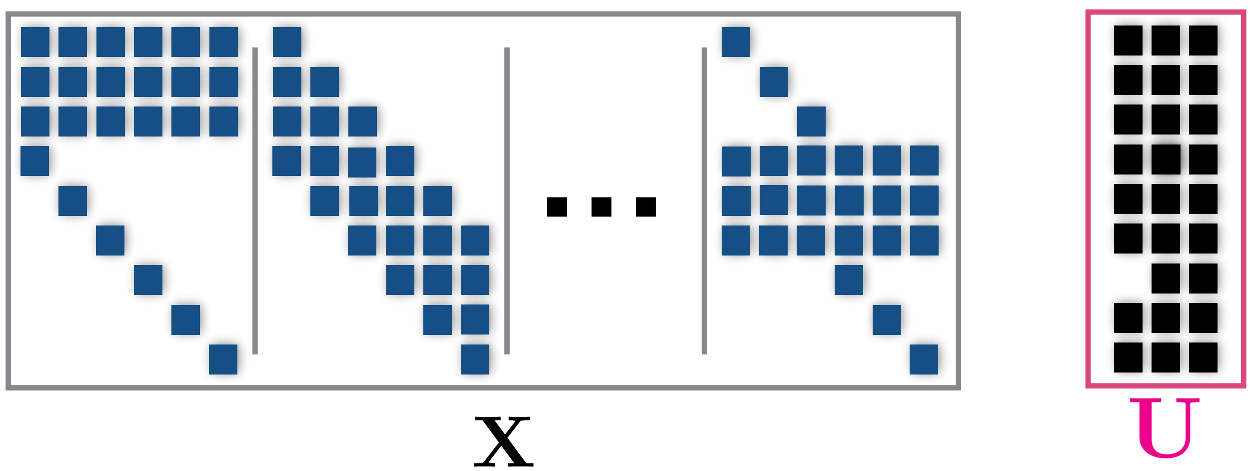
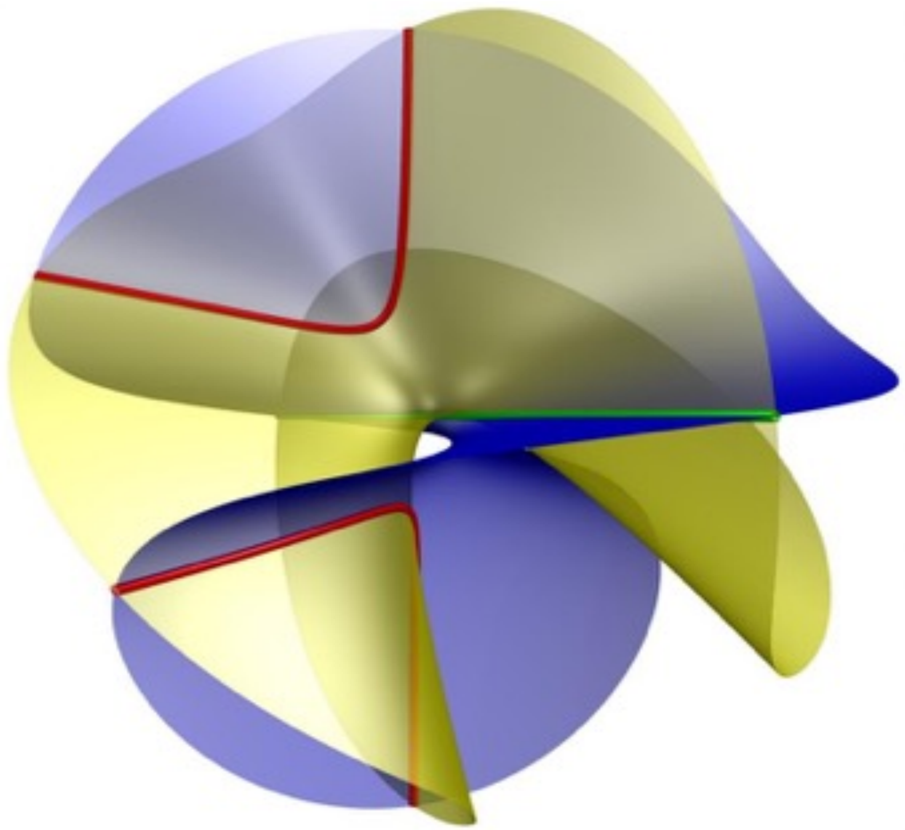


A flavor of our ideas

- If data is observed in *the right entries*
    - Polynomials are algebraically independent

- After this, use cool Algebraic Geometry tricks:
    - Polynomials are a regular sequence
    - Polynomials define a zero-dimensional variety
    - At most finitely many solutions
    - Unique solution (with a bit more work)

A flavor of our ideas



Each column with *r+1* entries imposes *1* restriction on what the subspace may be

Degree-*r* polynomial

$$\begin{bmatrix} \cdot & x_2 \\ y_1 & y_2 \\ z_1 & \cdot \end{bmatrix}$$

Entries observed in *the right places.*

↓

Polynomials are independent.

↓

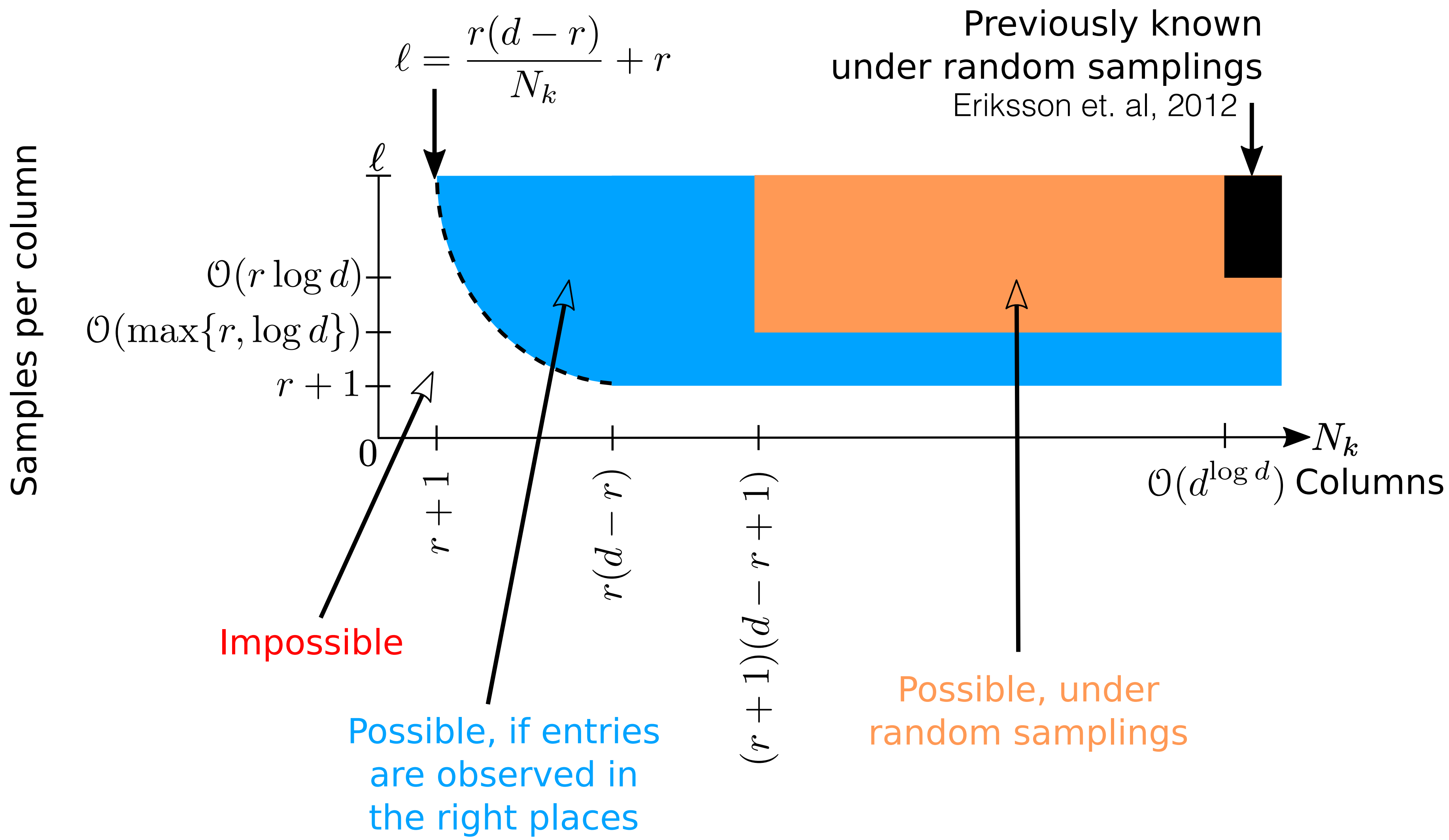
Completion is solution to polynomials.

A flavor of our ideas

Summary


WOW, AMAZING
PLEASE TELL
ME MORE

Can this be done? When? How?



Information-theoretic requirements

P.-A., Nowak, 2016

Can this be done? When? How?

Samples \ Computation	Good	Bad
Good	• EM [P.-A. Balzano, Nowak, 2014] • GSSC [P.-A. et. al, 2016] • MSC [P.-A. al, 2016] • SSC-EWZF [Wang et. al, 2016] • K-GROUSE [Balzano et. al, 2016]	Polynomials [P.-A. et. al, 2016]
Bad	HRMC [Eriksson et. al, 2012]	Who Cares

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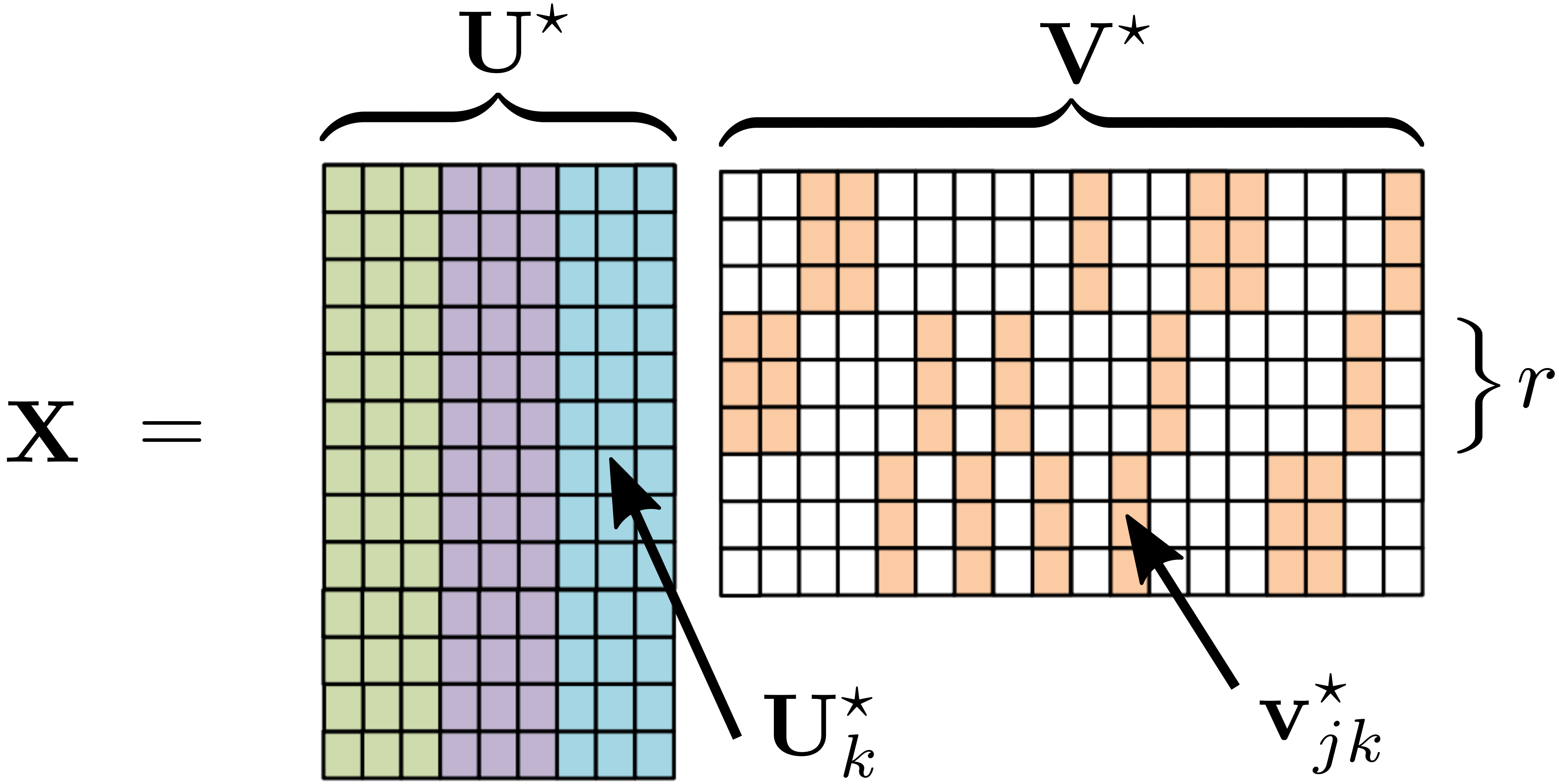
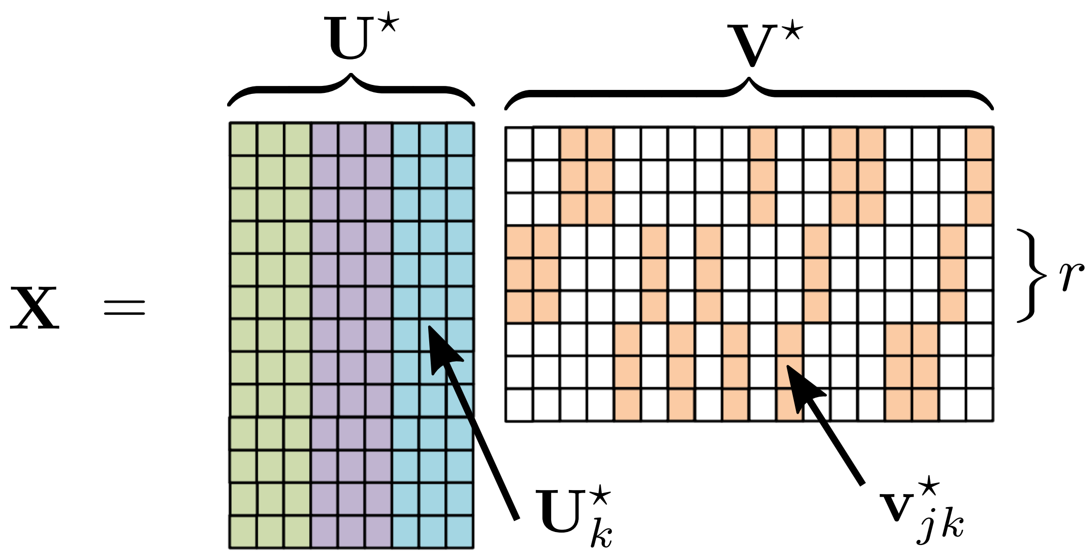


State-of-the-art Algorithms

P.-A. et. al, 2016



Algorithm 1: Group-Sparse Subspace Clustering

Input: $\mathbf{X}_\mathbf{\Omega}, K, r, \lambda$.

Initialize $\widehat{\mathbf{U}} \in \mathbb{R}^{d \times Kr}$ (e.g., using SSC-EWZF).

repeat

$$\widehat{\mathbf{V}} = \arg\min_{\mathbf{V}} \|\mathbf{\Omega}(\mathbf{X} - \widehat{\mathbf{U}}\mathbf{V})\|_F^2 + \lambda \sum_{j,k=1}^{N,K} \|\mathbf{v}_{jk}\|_2.$$

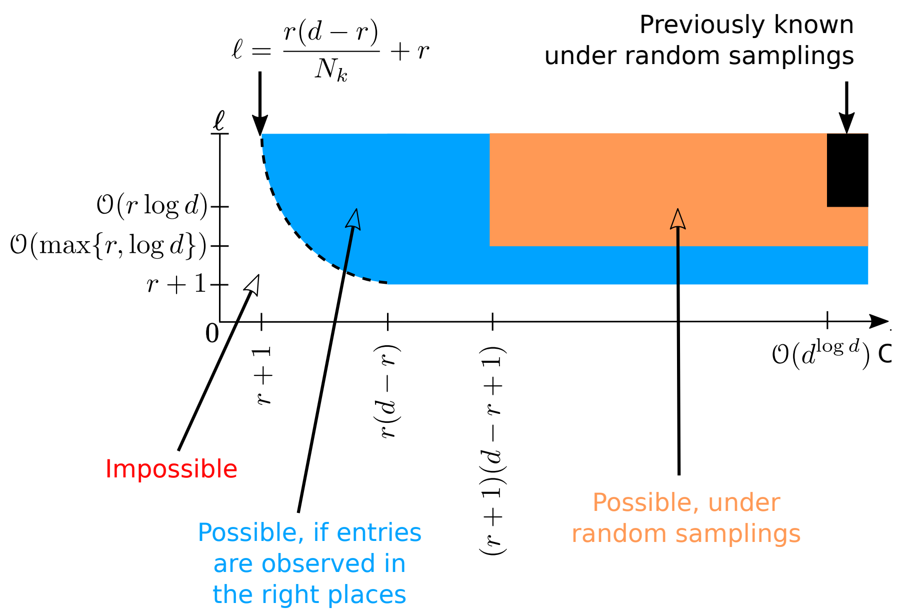
$$\widehat{\mathbf{U}} = \arg\min_{\mathbf{U} \,:\, \|\mathbf{U}\|_F \leq 1} \|\mathbf{\Omega}(\mathbf{X} - \mathbf{U}\widehat{\mathbf{V}})\|_F.$$

until convergence;

Output: $\widehat{\mathbf{U}}, \widehat{\mathbf{V}}$.

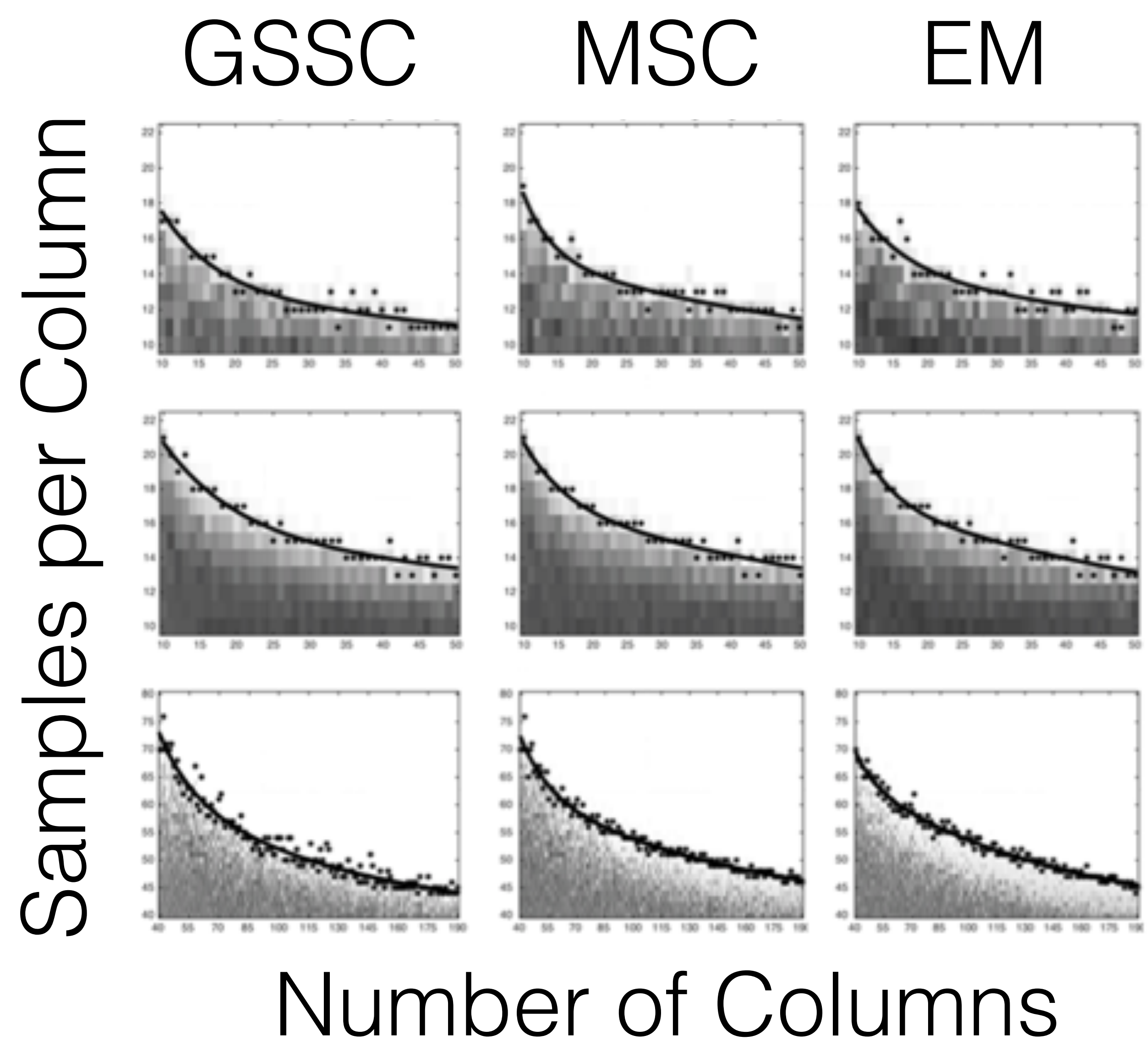
State-of-the-art Algorithms

P.-A. et. al, 2016



P.-A., Nowak, 2016

P.-A. et. al, 2016



Theory matches Practice

Can this be done? When? How?

Samples \ Computation	Good	Bad
Good	• EM [14] P.-A. et. al,, 2014 • GSSC [7] P.-A. et. al, 2016 • MSC [7] P.-A. et. al, 2016 • SSC-EWZF Wang et. al, 2016 • K-GROUSE Balzano et. al, 2016	Polynomials [10] P.-A. et. al, 2016
Bad	HRMC Eriksson et. al, 2012	Who Cares

Can this be done? When? How?

Samples \ Computation	**Good**	**Bad**
Good	• EM [14] P.-A. et. al,, 2014 • GSSC [7] P.-A. et. al, 2016 • MSC [7] P.-A. et. al, 2016 • SSC-EWZF Wang et. al, 2016 • K-GROUSE Balzano et. al, 2016 **Provable?**	Polynomials [10] P.-A. et. al, 2016
Bad	HRMC Eriksson et. al, 2012	Who Cares


Rob Nowak


Nigel Boston


Laura Balzano

Joint work with:


Becca Willett


Roummel Marcia


Steve Wright

Thank you!