Deterministic Conditions for Subspace Identifiability from Incomplete Sampling

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SILO
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Robert Nowak and Nigel Boston

Outline

- Problem Description
- Setup
- The Answer
- Sketch of the proof
- Application
- Conclusions

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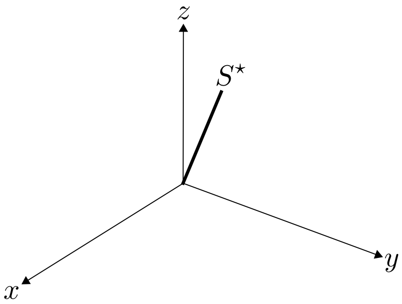
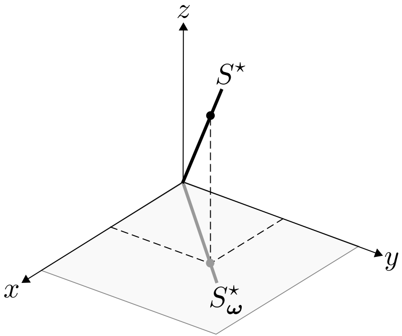
Problem description

$S^\star :=$ r-dimensional subspace of $\mathbb{R}^d$, $r < d$.



Problem description

$S_{\boldsymbol{\omega}}^{\star}$:= Projection of $S^{\star}$ onto a canonical subspace.



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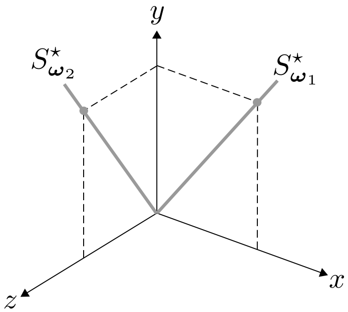

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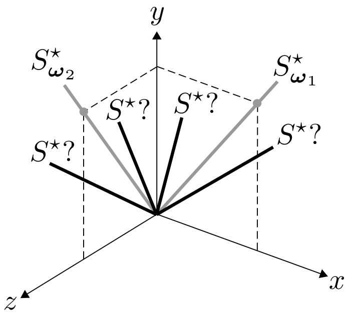
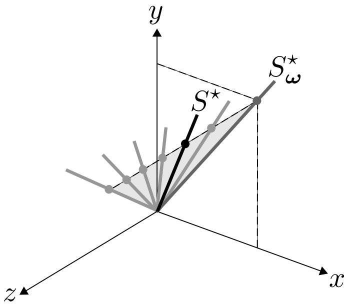


Can you uniquely determine $S^\star$ from this set of projections?

Problem description

Is this even possible?

There might be many subspaces that agree with the projections.



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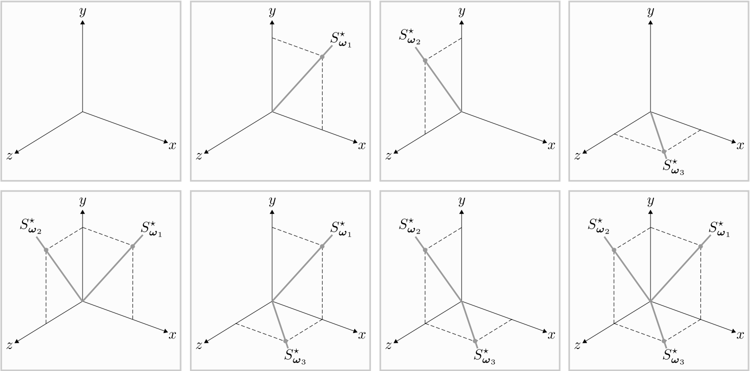
Well... it depends on which set of projections I give you.



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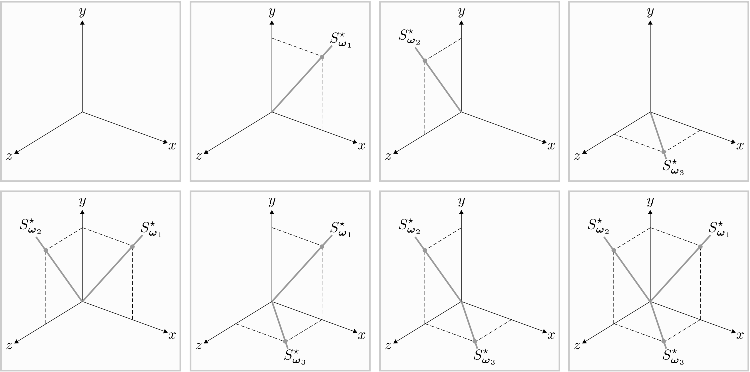

Can you tell which are *the good sets*?

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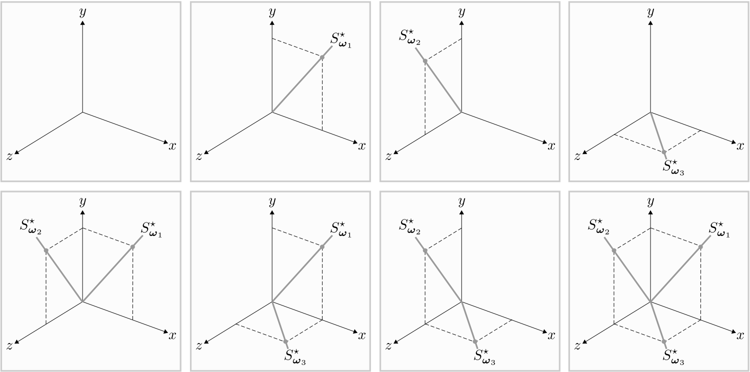

Can you tell which are *the good sets*?
This is what we answer here: which are *the good sets*.

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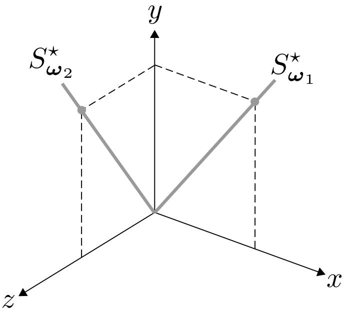
Setup

The columns of $\mathbf{\Omega}$ will index the given projections.



$$\mathbf{\Omega} \;=\; \begin{array}{c} \begin{matrix} \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Setup

In higher dimensions:

$$\boldsymbol{\Omega} = \begin{array}{c} \begin{matrix} \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \quad \Rightarrow \quad \left\{ \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ u_{41} & u_{42} \\ u_{51} & u_{52} \\ u_{61} & u_{62} \end{bmatrix} \right\}$$

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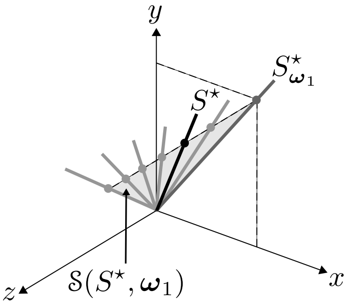

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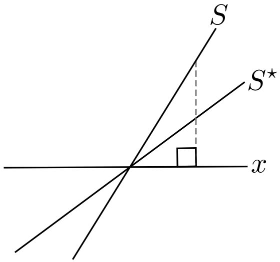
- The projection of $S^\star$ onto $\leq r$ canonical coordinates gives no information about $S^\star$.



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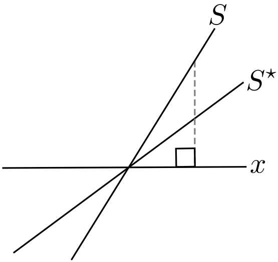

- $\Rightarrow$ Assume w.l.o.g. that all projections are onto $r + 1$ canonical coordinates.

Setup

- For any matrix $\mathbf{\Omega}'$ formed with a subset of the columns in $\mathbf{\Omega}$:

$$\mathbf{\Omega}' = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{n(\mathbf{\Omega}') \; := \; \#\text{columns}} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m(\mathbf{\Omega}') := \#\text{nonzero rows}$$

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- $d - r$ projections are *necessary*, so we will assume w.l.o.g.

$$n(\mathbf{\Omega}) = d - r.$$

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The Answer

Theorem (Pimentel-Alarcón, Nowak, Boston, '14)

For almost every $S^\star$*, with respect to the uniform measure over* $\mathrm{Gr}(r, \mathbb{R}^d)$*,* $S^\star$ *is the only subspace in* $\mathcal{S}(S^\star, \mathbf{\Omega})$ *if and only if for every matrix* $\mathbf{\Omega}'$ *formed with a subset of the columns in* $\mathbf{\Omega}$*,*

$$m(\mathbf{\Omega}') \;\geq\; n(\mathbf{\Omega}') + r.$$

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There is a set of measure zero of *bad* subspaces that we wouldn't identify.

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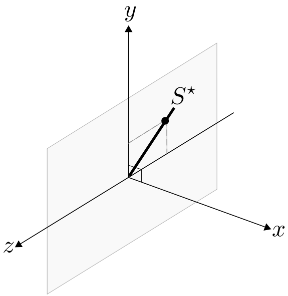

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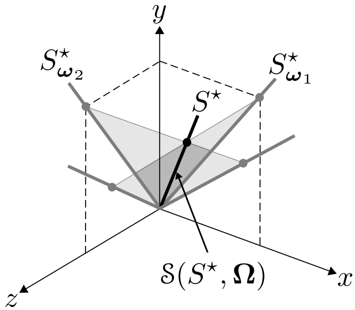
This is what we want!



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This is the answer!

Every subset of n columns of $\mathbf{\Omega}$ has at least $n + r$ nonzero rows.

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This is the answer!

Every subset of n columns of $\mathbf{\Omega}$ has at least $n + r$ nonzero rows.

$$\mathbf{\Omega} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Check: } \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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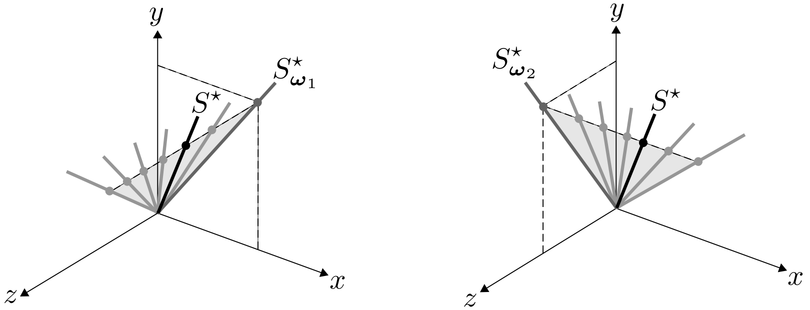
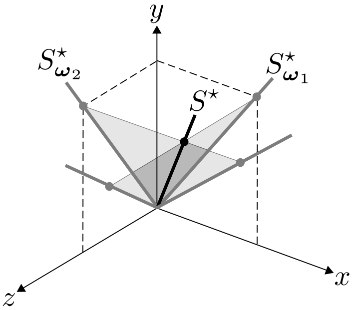
Sketch of the proof

We will find the subspaces that agree with *each* projection.



Sketch of the proof

Then find the intersection.



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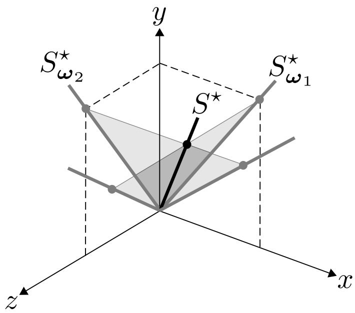
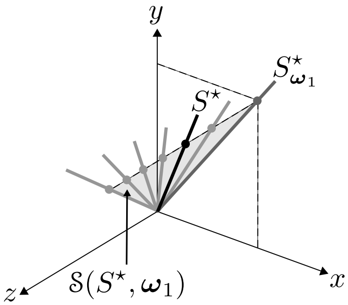


If the intersection only contains one subspace, then ;)

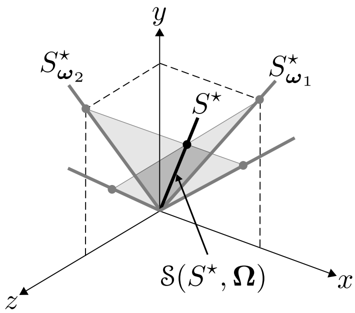
Sketch of the proof

$\mathcal{S}(S^\star, \boldsymbol{\omega}_i) :=$ Set of r-dimensional subspaces matching $S^\star$ on $\boldsymbol{\omega}_i$.



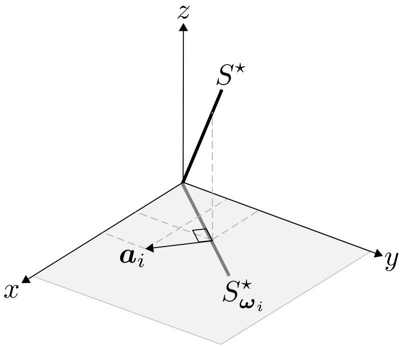
Sketch of the proof

$$\mathcal{S}(S^\star, \boldsymbol{\Omega}) = \bigcap_i \mathcal{S}(S^\star, \boldsymbol{\omega}_i).$$



Sketch of the proof

$\boldsymbol{a}_i :=$ Vector orthogonal to the i^{th} projection.



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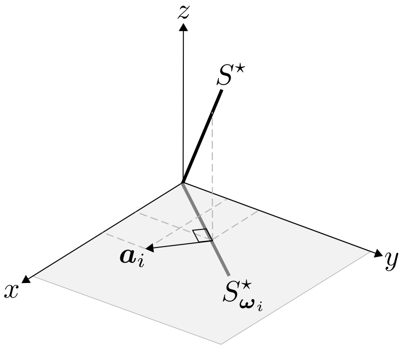

An entry in $\boldsymbol{a}_i$ is zero iff the corresponding entry in $\boldsymbol{\omega}_i$ is zero.

Sketch of the proof

One great thing:

- Every subspace in $\mathcal{S}(S^\star, \boldsymbol{\omega}_i)$ is orthogonal to $\boldsymbol{a}_i$.

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Cool! $\Rightarrow$

- Construct

$$\boldsymbol{A} = \left[\ \boldsymbol{a}_1 \ \middle|\ \cdots \ \middle|\ \boldsymbol{a}_N\ \right].$$

Sketch of the proof

One great thing:

- Every subspace in $\mathcal{S}(S^\star, \boldsymbol{\omega}_i)$ is orthogonal to $\boldsymbol{a}_i$.

Cool! $\Rightarrow$

- Construct

$$\boldsymbol{A} = \left[\begin{array}{c|c|c} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_N \end{array} \right].$$

- Every $S \in \mathcal{S}(S^\star, \boldsymbol{\Omega})$ must be contained in

$$\ker \boldsymbol{A}^\mathsf{T}.$$

Sketch of the proof

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- If $\dim\ker \boldsymbol{A}^\mathsf{T} = r$
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- If $\dim \ker \boldsymbol{A}^\mathsf{T} = r$
  $\Rightarrow$ Only $S^\star$ will agree with the projections. Moreover,

$$S^\star = \ker \boldsymbol{A}^\mathsf{T}$$

.

Sketch of the proof

- For any matrix $\boldsymbol{A}'$ formed with a subset of the columns in $\boldsymbol{A}$:

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- We want $\dim \ker \boldsymbol{A}^\mathsf{T} = r$, so $\boldsymbol{A}$ better have $d - r$ linearly independent columns.

Sketch of the proof

We know how to deal with $\boldsymbol{A}$ using linear algebra!

- Through some technical details:

Lemma (Pimentel-Alarcón, Nowak, Boston, '14)

For almost every $S^\star$, the columns of $\boldsymbol{A}$ are linearly dependent if and only if $m(\boldsymbol{A}') < n(\boldsymbol{A}') + r$ for some matrix $\boldsymbol{A}'$ formed with a subset of the columns in $\boldsymbol{A}$.

Sketch of the proof

The zero entries of $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ are in the same positions.

$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longleftrightarrow \quad \boldsymbol{A}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \\ 0 & 0 & a_{43} \end{bmatrix}$$

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Then

$$m(\boldsymbol{\Omega}') \geq n(\boldsymbol{\Omega}') + r \quad \Longleftrightarrow \quad m(\boldsymbol{A}') \geq n(\boldsymbol{A}') + r$$

$\square$

Sketch of the proof

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- Iff $S^\star$ is the only subspace in $\mathcal{S}(S^\star, \boldsymbol{\Omega})$. $\square$

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Low-Rank Matrix Completion (LRMC)

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- Given a subset of entries in a rank r matrix, exactly recover *all* of the missing entries.

$$\mathfrak{X}_{\mathbf{\Omega}} = \begin{bmatrix} 1 & \cdot & 3 & \cdot \\ 1 & 2 & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & 4 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathfrak{X}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

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- $\sim$ Identifying the subspace spanned by the columns, $S^{\star}$. Here

$$\hat{S} = \mathrm{span} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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- Maybe the real completion is:

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And the real subspace is

$$S^{\star} = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Application

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Known results e.g. (Candès and Recht, '09)

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What if these assumptions are not met? How can we validate a completion?

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Corollary (Pimentel-Alarcón, Nowak, Boston, '14)

Let the columns of $\mathfrak{X}$ *be drawn independently according to* μ*, an absolutely continuous distribution with respect to the Lebesgue measure on* $S^\star$*. Suppose* $\mathfrak{X}_{\mathbf{\Omega}}$ *can be partitioned into two sets of columns,* $\mathfrak{X}_{\mathbf{\Omega}_1}$ *and* $\mathfrak{X}_{\mathbf{\Omega}_2}$*, such that* $\mathbf{\Omega}_2$ *satisfies the conditions of the subspace identifiability theorem.*
Let $\hat{S}$ *be the output of running an LRMC algorithm on* $\mathfrak{X}_{\mathbf{\Omega}_1}$*.*
Then for almost every $S^\star$*, and almost surely with respect to* μ*,* $\mathfrak{X}_{\mathbf{\Omega}_2}$ *fits in* $\hat{S}$ *if and only if* $\hat{S} = S^\star$*.*

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Let the columns of $\mathfrak{X}$ be drawn independently according to μ, an absolutely continuous distribution with respect to the Lebesgue measure on $S^\star$. Suppose $\mathcal{X}_\Omega$ can be partitioned into two sets of columns, $\mathcal{X}_{\Omega_1}$ and $\mathcal{X}_{\Omega_2}$, such that Ω_2 satisfies the conditions of the subspace identifiability theorem.

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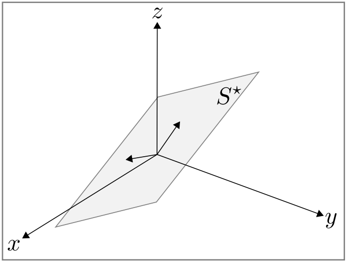

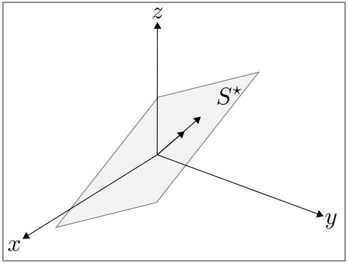

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Outline

- Problem Description ✓
- Setup ✓
- The Answer ✓
- Sketch of the proof ✓
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Now we know that:

- It is possible to uniquely identify an r-dimensional subspace $S^\star$ from its projections onto $\mathbf{\Omega}$.
- If and only if every subset of n columns of $\mathbf{\Omega}$ has at least $n+r$ nonzero rows.
- Whence $S^\star = \ker \boldsymbol{A}^\mathsf{T}$.

Thanks.