CS 4980/6980: Data Science © Spring 2018

Lecture 6: Logistic Regression

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6.1 Introduction

Logistic regression is the regression analysis which is implemented when the dependent variable is binary. Like all regression analyses, the logistic regression is a predictive analysis. Logistic regression is used to describe data and to explain the relationship between one dependent binary variable and one or more nominal, ordinal, interval or ratio-level independent variables. It is applied when an event has only two results, such as *True/False* or *Pass/Fail*.

Now, we introduce concepts of the Logistic Regression.

- Generalized Logistic Models.
- Predict variable of interest for rest of data.
- Study log(odds).
- Study the "Likelihood" of the data.
- Difference between probability and likelihood.
- Model the probabilities of variable of interest as a function of some explanatory variables.
- Find argument that finds maximum likelihood.

6.2 Logistic Regression

First, we would like to show a brief introduction to probability

$$y \in \{0, 1\}$$

In the above equation, y is a categorical variable, which means that output can take only two values; "0" or "1", that represents outcomes such as True/False, Win/Lose, or Dead/Alive. If random variable y has the following equation:

$$P(y = 1) = p$$

$$P(y = 0) = 1 - p$$

$$P(y,k) = \begin{cases} \text{p} \quad if \;\; k = 1, \\ \text{1-p} \;\; if \;\; k = 0 \end{cases}$$

For $0 < p < 1$, y is called *Bernoulli* random variable and it expressed as

$$Y \sim Bernoulli(p)$$

Probability is used to compare two events. Assume p is the event of a success and $1 - p$ as the event of a failure. Then take probability of success and divide it by probability of a failure to find its **odds**

$$odds := \frac{p}{1-p}$$

Odds by definition, is the extent to which an event is likely to occur. It is measured by the real number of the favorable cases possible. So in the above equation, p must be between 0 and 1.

Now, taking the Logarithm for probability of odds

$$log(\frac{p}{1-p})$$

Then formally, the model of logistic regression can be represented as:

$$log(\frac{p}{1-p}) = \beta_1 x_i + \beta_2 x_{ii} + ... + \beta_d x_d$$

We have taken Log of the odds ratio. The Log of odds are given by the linear combination of the independent variable and the parameters. In a Logistic Model, the parameter of the model beta and X are crucial to determine the odds of a success. The larger the number is, the higher the probability of success.

6.3 Probability

Here, we are calculating the probability based on logs of the odds ratio. Now, to get the value of probability, Exponential function is applied to solve for p.
Applying Exponential function on log odds ratio we get :

$$\frac{p}{1-p} = e^{(\beta^T X)}$$

$$p = (1-p)e^{(\beta^T X)}$$

$$p = e^{(\beta^T X)} - pe^{(\beta^T X)}$$

$$p + pe^{(\beta^T X)} = e^{(\beta^T X)}$$

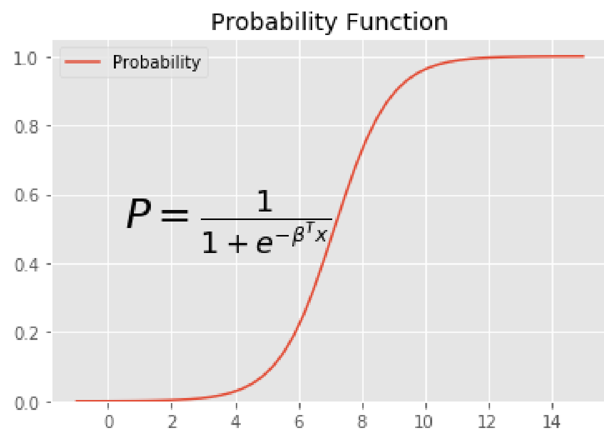
$$p(1 + e^{(\beta^T X)}) = e^{(\beta^T X)}$$

$$p = \frac{e^{(\beta^T X)}}{1+e^{(\beta^T X)}}$$

The expression of probability can be further simplified as follows. Now let's divide Numerator and Denominator and bottom by : -

$$e^{(\beta^T X)}$$

$$p = \frac{1}{1+e^{(-\beta^T X)}}$$



P represents the probability of 1 and e is the base of the natural logarithm. Finally, beta transpose T and feature vector x are the parameters of the model.

Now, let's take an example of Titanic. In the following equations survival and perished cases are demonstrated. Survival case :- odds of survival

$$(\frac{1}{1+e^{(-\beta^T X)}}) > (1 - \frac{1}{1+e^{(-\beta^T X)}})$$

Here y=1

Perished case :- odds of perished

$$(\frac{1}{1+e^{(-\beta^T X)}}) < (1 - \frac{1}{1+e^{(-\beta^T X)}})$$

Here y=0

Till here, we have completed the setup for Logistic Regression, now our next goal is to find β

6.4 Likelihood

This section will introduce **Likelihood** function in Logistic Regression. Concept of Probability is similar to the concept of **Likelihood** , in which yi is the random variable and Yi is the observed sample.

$$L(p|Y_i) = P(y_i|p)$$

Now let's take an example to represent Likelihood and Probability.

Example 6.1. For Probability : -

$$f(x|a, b) = ax + b$$

And for Likelihood : -

$$f(a|x, b) = ax + b$$

Example of Likelihood Vs Probability

$$x \sim N(\mu, \sigma^2)$$

$$P(x) = \frac{1}{\sqrt{(2\pi)}\sigma} e^{(\frac{-1}{2})(\frac{x-\mu}{2})^2}$$

$$P(x|\mu, \sigma)$$

Here x is the variable of interest

$$L(x) = \frac{1}{\sqrt{(2\pi)}\sigma} e^{(\frac{-1}{2})(\frac{x-\mu}{2})^2}$$

$$L(\mu|x, \sigma)$$

Here the variable of interest is μ

$$\text{Probability of the } i^{th} \text{ sample:-}$$

$$P(y_i) = P^{(y_i)}(1-p)^{(1-y_i)}$$

$$\text{Likelihood of the } i^{th} \text{ sample:-}$$

$$L(P|Y_i) = P^{(Y_i)}(1-P)^{(1-Y_i)}$$

Now, as we already know

$$P = \frac{1}{1+e^{(-\beta^T X)}}$$

Hence, substituting the values in the above equation.

$$L(\beta|Y_i) = (\frac{1}{1+e^{(-\beta^T X_i)}})^{(Y_i)}(\frac{1}{1+e^{(-\beta^T X_i)}})^{(1-Y_i)}$$

And as

$$P(y_1, y_2, \ldots, y_n)^p = \prod_{i=1}^{N} P(y_i)$$

$$= \prod_{i=1}^{N} p^{(y_i)}(1-p)^{(1-y_i)}$$

Now Similarly for Likelihood

$$L(P|Y_1, Y_2, \ldots, Y_N) = \prod_{i=1}^{N} L(P|Y_i)$$

$$= \prod_{i=1}^{N} P^{(Yi)}(1-P)^{(1-Yi)}$$

$$= \prod_{i=1}^{N} (\frac{1}{1+e^{(-\beta^T X)}})^{(Yi)} (1 - (\frac{1}{1+e^{(-\beta^T X)}}))^{(1-Yi)}$$

We will take derivative of the above equation and solve for β as it is our variable of interest. Now applying log to the above equation we get,

$$log(L(P|Y_1, Y_2, \ldots, Y_N)) =: l(P|Y)$$

$$log[\prod_{i=1}^{N} (\frac{1}{1+e^{(-\beta^T X_i)}})^{(Y_i)} (1 - (\frac{1}{1+e^{(-\beta^T X_i)}}))^{(1-Y_i)}]$$

$$\sum log((\frac{1}{1+e^{(-\beta^T X_i)}})^{(Y_i)} + log((1 - (\frac{1}{1+e^{(-\beta^T X_i)}}))^{(1-Y_i)})$$

$$l(P|Y) = \sum_{i=1}^{N} Y_i log(\frac{1}{1+e^{(-\beta^T X_i)}}) + (1 - Y_i) log((1 - (\frac{1}{1+e^{(-\beta^T X_i)}}))$$

or

$$l(\beta|Y) = \sum_{i=1}^{N} Y_i log(\frac{1}{1+e^{(-\beta^T X_i)}}) + (1 - Y_i) log((1 - (\frac{1}{1+e^{(-\beta^T X_i)}}))$$

Now, we have to find the best value of β

$$\hat{\beta} := argmax(l(\beta|y))$$

$$\text{Where } \ \beta \in R^D$$

6.5 Conclusion

This lecture shows logic behind Logistic Regression. Like the basic Regression technique i.e. Linear Regression, Logistic Regression provides coefficient β, which measures each variable as information as well as odds ratio that represents probability of success. The goal is to correctly predict the outcome for each case using this model.